BMOS Mentoring Scheme 2013 - 14 (Intermediate Level)

Sheet 1 - Example Solutions and Comments

I have 10 pairs of socks in my drawer, but all of the socks are separate. Each pair is a different colour. I shut my eyes and pull out some socks. How many must I take out to **guarantee** that I get a pair? How many more do I need in order to get a second pair?

Answer: 11.

If I take out only 10 socks, I could get one from each pair and so not have a pair. But if I take out 11, then I must get a pair. To get a second pair, I need to remove only one more sock.

This is a good example of the *Pigeonhole Principle* (also called *Dirichlet's Principle*), which states that if I have n pigeonholes and $n+1$ pigeons in the pigeonholes, then at least one pigeonhole must contain at least 2 pigeons. (Note that I don't know that every pigeonhole contains a pigeon: the pigeons could all be in the same hole!) Despite its apparent simplicity, this is a very useful result

2. Find the lowest common multiple and highest common factor of 47880 and 67620.

Answer: lowest common multiple 7708680, highest common factor 420

To find the lowest common multiple, we first factorise the numbers into primes. We have $47880 = 2^3 \times 3^2 \times 5 \times 7 \times 19$ and $67620 = 2^2 \times 3 \times 5 \times 7^2 \times 23$. We then compare these to find that the least common multiple of the two is $2^3 \times 3^2 \times 5 \times 7^2 \times 19 \times 23 = 7708680$ (for each prime, we take the higher power to which it appears in one of the numbers). Convince yourself that this really is a multiple of both numbers and why it is the least such.

Having found the prime factorisations, it is quite easy to find the highest common factor: this time we take the lower power of each prime that appears in the two numbers, to get $2^2 \times 3 \times 5 \times 7 = 420$. Convince yourself that this is the highest common factor.

There is another method of finding the highest common factor, that does not involve finding the prime factorisation, called *Euclid's algorithm*. This is useful, because it is normally quite hard to find prime factorisations. I shall use the numbers above to show you how it works.

We divide 67620 by 47880 and get a remainder: $67620 = 1 \times 47880 + 19740$.

Now we divide 47880 by 19740: $47880 = 2 \times 19740 + 8400$.

Keep going: $19740 = 2 \times 8400 + 2940$.

$8400 = 2 \times 2940 + 2520$.

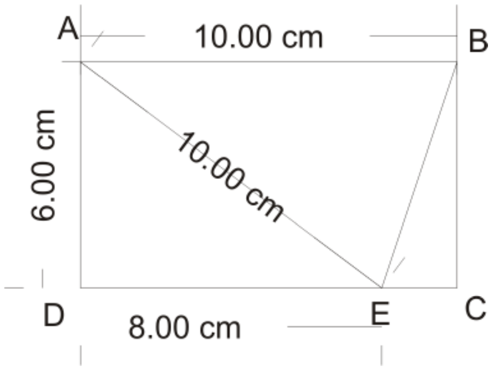
$2940 = 1 \times 2520 + 420$.

$2520 = 6 \times 420 + 0$.

Now let d be the highest common factor of 67620 and 47880. From the first equation, we know that d divides 19740 (because it divides 67620 and 1×47880). Now d divides 47880 and 19740, so from the second equation we know that it divides 8400. And so on, all the way down, until we find that d divides 420. Also, 420 divides 420 and 2520, so from the penultimate equation we know that it divides 2940, so it divides 8400, and so on, until we find that 420 divides both 47880 and 67620. So 420 is a common factor, and the highest common factor divides 420, so the highest common factor *is* 420 (which is what we found earlier).

3. $ABCD$ is a rectangle. Sides AB and CD have length 10cm, and sides BC and DA have length 6cm. A point E is marked on side CD such that it is 2cm from C. Show that $\angle AEB = \angle BEC$. ($\angle ABC$ means the angle between AB and BC, for example.)

We can draw in the line segment AE and use Pythagoras' Theorem to find its length: 10cm.



Now the triangle ABE is isosceles (since sides AB and AE each have length 10cm), so $\angle ABE = \angle AEB$. By considering properties of parallel lines (or noting that they are both $90° - \angle EBC$), we find that $\angle ABE = \angle BEC$, and so $\angle AEB = \angle BEC$.

4. 12 has 6 factors: 1, 2, 3, 4, 6, 12. Find a criterion for a number to have an odd number of factors. (That is, find some simple property that is held by all numbers with an odd number of factors, but is not held by any number with an even number of factors.)

Answer: A whole number n has an odd number of factors if and only if n is a perfect square.

I think that the way to start this question is with some experimentation. Hopefully you will then spot a pattern, and can go on to prove it.

We can pair off factors of n (except for its square root) such that each pair multiply to give the number. (For example, for 12 we can pair off 1 and 12, 2 and 6, and 3 and 4.) So n has an odd number of factors if and only if it has a factor that is a square root, as all its other factors will pair off. So n has an odd number of factors if and only if it is a perfect square.

5. Find all real numbers x such that $\sqrt{13 - 4x} = 2x + 1$.

Answer: $x = 1$ is the only solution.

The obvious thing to do at this stage is to square both sides, to get rid of the square root. However, squaring equations adds extra solutions. (For example, $x = 1$ clearly only has one solution, but squaring both sides we get $x^2 = 1$, which has two solutions.) This is fine: what we do is square the equation, work through to get values of x, and then check which ones satisfy the original equation. It is vital that you remember to check that the values work!

Squaring, we get $13 - 4x = 4x^2 + 4x + 1$.

Rearranging this, we get $4x^2 + 8x - 12 = 0$.

Dividing by 4 and factorising, this becomes $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$. So $x = -3$ or $x = 1$.

Now we check these in the original equation. If $x = -3$, then $\sqrt{13 - 4x} = \sqrt{13 + 12} = 5$, but $2x + 1 = -6 + 1 = -5$. So $x = -3$ is not a solution of the original equation. If $x = 1$, then $\sqrt{13 - 4x} = \sqrt{13 - 4} = 3$, and $2x + 1 = 2 + 1 = 3$, so $x = 1$ is a solution. So the only real number that satisfies the equation is $x = 1$.

6. (i) For a, $b \geq 0$, prove that $\frac{a+b}{2} \geq \sqrt{ab}$. When can equality occur?

($\frac{a+b}{2}$ is the arithmetic mean of a and b; $\sqrt{ab}$ is the geometric mean.)

(ii) For a, $b \geq 0$, prove that $\sqrt{ab} \geq 1/\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right)$. When can equality occur?

($1/\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right)$ is the harmonic mean of a and b.)

When proving things like inequalities, it is vital to write out one's argument properly. When trying initially to prove it, one almost certainly starts with what one has to prove, then manipulates it until it becomes something that we know is true. This is a sensible way to approach such questions. However, this does not provide a proper mathematical proof: a proof must start with something that we know to be true, and then work from there one step at a time until the final result is reached. In practice, this means that you work out the proof, and then write it out the other way round, checking that the steps are still valid. A "proof" will gain almost no credit if it is in the wrong order!

(i) Since a and b are non-negative (meaning ≥ 0), we know that $a = (\sqrt{a})^2$ and $b = (\sqrt{b})^2$. Now $(\sqrt{a}-\sqrt{b})^2 = (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 = a + b - 2\sqrt{ab}$. Note that $(\sqrt{a}-\sqrt{b})^2 \geq 0$ (squares are always non-negative: this is an important fact to remember). So $a + b - 2\sqrt{ab} \geq 0$, so $a + b \geq 2\sqrt{ab}$, so $\frac{a+b}{2} \geq \sqrt{ab}$. The inequality came from the fact about squares being non-negative; $x^2 = 0$ if and only if $x = 0$, so we have equality if and only if $\sqrt{a} - \sqrt{b} = 0$, that is, if and only if $a = b$.

An alternative approach would be to start from $(a-b)^2 \geq 0$. Multiplying out, we get $a^2+b^2 \geq 2ab$, so, adding $2ab$ to both sides, $a^2 + 2ab + b^2 \geq 4ab$. Now we can factorise the left-hand side to get $(a+b)^2 \geq 4ab$, so taking the square root, $a + b \geq 2\sqrt{ab}$ and so $\frac{a+b}{2} \geq \sqrt{ab}$. This is true, as long as we know that $x^2 \geq y^2$ and x, $y \geq 0$ means that $x \geq y$. Finding when equality occurs is almost identical to the previous argument.

(ii) This does not really require any more work. Let $c = 1/a$ and $d = 1/b$. Now c, $d \geq 0$, so we may apply the inequality from part (i) to find $\frac{c+d}{2} \geq \sqrt{cd}$. Writing this in terms of a and b, we have $\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \sqrt{\frac{1}{ab}}$. Now we may multiply by $\sqrt{ab}$ (as it is positive) and divide by $\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ (as it is also positive) to get $\sqrt{ab} \geq 1/\left(\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right)$. By part (i), equality occurs if and only if $c = d$, that is, if and only if $1/a = 1/b$, that is, if and only if $a = b$.

7. How many ways are there of arranging the letters *abc*? What about *abcd*?
(*abc* is different from *bac*, for example.)
How many ways are there of arranging the letters *camera*? What about *banana*?

Answers: 6, 24, 360, 60

There are 6 ways of arranging *abc*: there are 3 possible letters to go first, then when we've chosen that there are 2 possibilities for the second place, and then the third one is fixed, so there are $3 \times 2 \times 1 = 6$ ways.

Similarly, there are $4 \times 3 \times 2 \times 1 = 24$ ways of arranging *abcd*. In general, there are $n \times (n-1) \times \ldots \times 3 \times 2 \times 1 = n!$ ways of arranging n distinct letters. ($n!$ is called n *factorial*, and is defined to be $n \times (n-1) \times \ldots \times 3 \times 2 \times 1$).

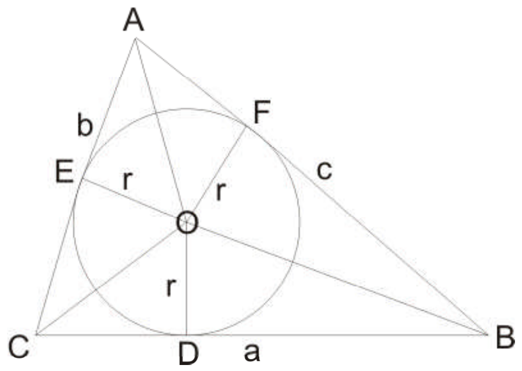
There are $6! = 720$ ways of arranging 6 letters, so we might think that this is the number of ways of arranging the letters *camera*. But two of the letters in *camera* are the same, so we need to allow for this. If they were different, say a_1 and a_2, then ca_1mera_2 and ca_2mera_1 would be different, and this is what we are counting in the 6!. But this means that we have counted twice as many arrangements as we should have done, because we can switch a_1 and a_2. So there are really $6!/2 = 720/2 = 360$ arrangements.

banana is slightly more complicated still. Again, if all of the letters were different, there would be 720 arrangements. We can allow for the 2 ns as above, to get 360 arrangements. But there are also

3 as that we need to consider. There are $3 \times 2 \times 1 = 6$ ways of arranging the three as, so we need to divide by this. So there are $6!/(2!3!) = 720/(2 \times 6) = 60$ arrangements.

8. The *incircle* of a triangle is the circle inside the triangle that is tangent to all three sides (it just touches each side).
Let ABC be a triangle with side lengths a, b, c and area $[ABC]$. Let r be the radius of the incircle of the triangle. Show that $[ABC] = \frac{1}{2}r(a+b+c)$.

The first thing to do, as always with geometry questions, is to draw a large, clear diagram.



There are three radii of the incircle marked on this diagram; they meet the sides of the triangles at right angles, because the incircle is tangent to the sides of the triangle. (This is a standard circle theorem; you should look up the circle theorems if you are not yet familiar with them. They occur in the GCSE syllabus.) Now the area of triangle ACO is $\frac{1}{2}br$, the area of CBO is $\frac{1}{2}ar$ and the area of BAO is $\frac{1}{2}cr$. These three triangles together combine to give the whole triangle, so $[ABC] = \frac{1}{2}(ar + br + cr) = \frac{1}{2}r(a+b+c)$.

The quantity $\frac{1}{2}(a+b+c)$ is called the *semi-perimeter* (it is half the total perimeter of the triangle) and is often denoted by s, so we could write $[ABC] = rs$.