BMOS MENTORING SCHEME (Intermediate Level)
May 2013 (Sheet 8)
Hints and Solutions for Mentors and Teachers

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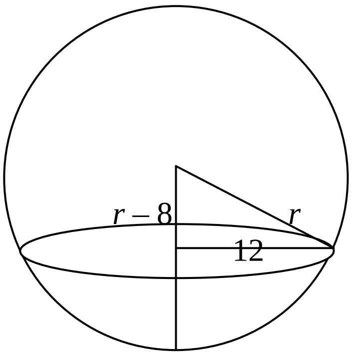
1. A ball is floating on a pond when the water freezes. When the ball is removed without breaking the ice, it leaves a hole 24cm across and 8 cm deep. What is the radius of the ball?

If we let the radius be r, then we have a right-angled triangle as shown with sides r, 12, and $r - 8$.
By Pythagoras,

$$(r-8)^2 + 12^2 = r^2$$

so $r^2 - 16r + 64 + 144 = r^2$
Hence $16r = 208$
giving $r = 13$ cm.



2. A sequence of integers $a_1, a_2, a_3, ...$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2013 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

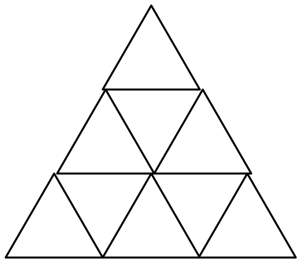
Maybe it would be helpful for mentees to begin to get into this problem by choosing a couple of random numbers for the first two terms of the sequence and see what happens. Let's try 2 and 5.
The sequence starts 2, 5, 3, -2, -5, -3, 2, 5, and we now see that it will repeat (but note this is only deducible when we have the pair 2, 5 repeating. It is not sufficient just to have the 2.
So we now need to prove the obvious conjecture by doing: $a, b, b - a, -a, -b, a - b, a, b, ...$ so we see that the sequence does indeed repeat every 6 terms, and moreover the sum of these 6 terms is zero.
So the sum of the first six terms, $S_6 = 0 = S_{12} = S_{18}$ etc.
So since $1492 \equiv 4 \pmod 6$, $S_{1492} = S_4 = 2b - a = 1985$,
and since $1985 \equiv 5 \pmod 6$, $S_{1985} = S_5 = b - a = 1492$.
Subtracting these gives $b = 493$ and hence $a = -999$.
So since $2013 \equiv 3 \pmod 6$, $S_{2013} = S_3 = 2b = 986$.

3. If (x, y) is a solution to the system $xy = 6$ and $x^2y + xy^2 + x + y = 63$, find the value of $x^2 + y^2$.

This just needs a bit of messing around with the equations, so mentees need to be ready to do this.
We note that the second equation can be rearranged to $xy(x + y) + (x + y)$ and since $xy = 6$, we have $7(x + y) = 63$, so $x + y = 9$. Now since $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 12$,
we have $x^2 + y^2 + 12 = 81$, and hence $x^2 + y^2 = 69$.

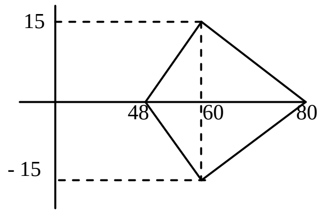
> *4. 100 spiders are crawling on a table shaped like an equilateral triangle, 3 metres per side. Prove that somewhere on the table there are 12 spiders all within 1 metre of each other.*

The key idea here is to split the triangle into 9 equilateral triangles with side 1 m. Now if each mini-triangle contains 11 spiders, we will have 99 spiders.
As soon as we place a 100^{th} spider on the table, at least one mini-triangle must contain 12 spiders or more. (The name for this argument is the Pigeon-Hole Principle.) So there must be 12 spiders somewhere on the table, all within 1 metre of each other.



> *5. Find the area enclosed by the graph whose equation is* $|x-60|+|y|=\left|\frac{x}{4}\right|$.

Mentees just need to keep their wits about them, and consider cases:
If $x > 60$, the equation simplifies to $x - 60 + |y| = {}^x/_4$, so $|y| = 60 - ¾x$, whilst it is non-negative, since it has to equal $|y|$. When $x = 60$, $|y| = 15$, so $y = \pm 15$ and when $x = 80$, $|y| = 0$, so $y = 0$.
The expression is linear, so these represent two line segments.
If $0 < x < 60$, the equation simplifies to $60 - x + |y| = {}^x/_4$, so $|y| = {}^{5x}/_4 - 60$, providing this is not negative. The value of x which makes $y = 0$ is 48 and when $x = 60$, $y = \pm 15$ as before.
If $x < 0$, the equation gives $60 - x + |y| = -{}^x/_4$, so $|y| = ¾x - 60$, which is always negative, so there are no possible values of y if $x < 0$.
So the graph consists of four line segments, forming a kite as shown.



So the area = 32 × 15 = 480.

> *6. There are 100 soldiers in a detachment, and every evening three of them are on duty. Can it happen that after a certain period of time each soldiers has shared duty with every other soldier exactly once?*

Consider an arbitrary soldier. If he is to share duty with all the other soldiers exactly once, then he must share duty with 99 soldiers. However each night he shares duty with two soldiers. So if these are all different, then the total number of soldiers he has shared duty with will always be even. Therefore he cannot ever have shared duty with exactly 99 others.

> *7. Find all positive integer pairs (a, b) such that* $4^a + 4a^2 + 4 = b^2$.

It is not clear how to tackle such a problem as this. One way is to use modular arithmetic – I don't know if this can be made to work here. Another way is to try to get some bounds on either a or b so as to reduce the search to a finite number of cases. This approach will work here as follows:

The equation implies that b^2 is even and greater than 4^a, which means that b is even and greater than 2^a, that is $b \geq 2^a + 2$, and

$$4^a + 4a^2 + 4 = b^2 \geq (2a+2)^2 = 4^a + 4\times 2^a + 4$$

which gives $a^2 \geq 2^a$, and consequently $a \leq 4$.
So we only have four possible values for a. If we substitute the possible values a = 1, 2, 3, 4 into the original equation we find there are exactly two solution pairs for (a, b), namely (2, 6) and (4, 18).

8. Find all pairs of prime numbers (p, q) which satisfy the equation

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{pq}=\frac{1}{n}\quad , \quad \textit{where } n \textit{ is a natural number.}$$

The equation $\Rightarrow \dfrac{q+p+1}{pq}=\dfrac{1}{n}$ so $q+p+1=\dfrac{pq}{n}$.

Since $q + p + 1$ is a natural number, n divides pq.

And since p and q are prime, the only possibilities are: $n = 1$, $n = p$, $n = q$ or $n = pq$.

If $n = 1$, $q+p+1=pq \Leftrightarrow q+1=pq-p=p(q-1) \Leftrightarrow \dfrac{q+1}{q-1}=p \Leftrightarrow 1+\dfrac{2}{q-1}=p.$

Thus since p is a natural number, $q - 1$ must divide into 2. The only possibilities are therefore $q = 2$ or $q = 3$.

If we substitute $n = 1$ and $q = 2$ into the equation, we get $p = 3$, which is therefore a solution.

If we substitute $n = 1$ and $q = 3$ into the equation, we get $p = 2$, which is therefore another solution.

If $n = p$ or q: Since the equation is symmetric with respect to p and q, we can assume without loss of generality that $n = p$.

Thus $q+p+1=\dfrac{pq}{p}=q \Leftrightarrow p=-1$, but since p is a prime number, this is not possible.

Thus n cannot equal either p or q.

If $n = pq$, $q+p+1=\dfrac{pq}{pq}=1 \Leftrightarrow q=-p$, but this implies that either q or p is negative, which cannot be the case. Thus n cannot be equal to pq.

Therefore the only two solutions are:

$n = 1$, $p = 2$, $q = 3$

$n = 1$, $p = 3$, $q = 2$.

I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to mentoring@ukmt.org is of course also very welcome.