UKMT MENTORING SCHEME (Intermediate Level)
May 2013 (Sheet 8)
Questions

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1. A ball is floating on a pond when the water freezes. When the ball is removed without breaking the ice, it leaves a hole 24cm across and 8 cm deep. What is the radius of the ball?

2. A sequence of integers $a_1, a_2, a_3, ...$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. (i.e. each term is the previous one minus the one before that). What is the sum of the first 2013 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?

3. If (x, y) is a solution to the system $xy = 6$ and $x^2y + xy^2 + x + y = 63$, find the value of $x^2 + y^2$.

4. 100 spiders are crawling on a table shaped like an equilateral triangle, 3 metres per side. Prove that somewhere on the table there are 12 spiders all within 1 metre of each other.

5. We define $|x|$ to take the value x when $x \geq 0$, and the value $-x$ when $x < 0$. ($|x|$ is called the "modulus of x"). Find the area enclosed by the graph whose equation is:

$$|x-60|+|y|=\left|\tfrac{x}{4}\right|$$

6. There are 100 soldiers in a detachment, and every evening three of them are on duty. Can it happen that after a certain period of time each soldier has shared duty with every other soldier exactly once?

7. Find all positive integer pairs (a, b) such that $4^a + 4a^2 + 4 = b^2$.

8. Find all pairs of prime numbers (p, q) which satisfy the equation

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{pq}=\frac{1}{n}\quad ,\qquad \text{where } n \text{ is a natural number.}$$

Deadline for receipt of solutions: 31st May 2013