BMOS MENTORING SCHEME (Intermediate Level)
April 2013 (Sheet 7)
Hints and Solutions for Mentors and Teachers

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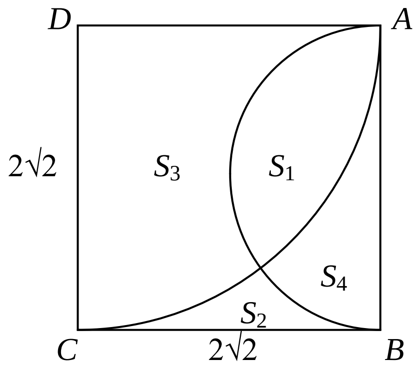
1. The square shown has side $2\sqrt{2}$. A semicircle diameter AB is shown, and also a quadrant centre D, passing through A and C. The areas of the two regions shown are S_1 and S_2. Calculate $S_1 - S_2$.



If we label the other regions S_3 and S_4 as shown, we have the area of the quadrant,

$$S_1 + S_3 = \tfrac{1}{4}\pi\,(2\sqrt{2})^2 = 2\pi$$

and the area of the semicircle,

$$S_1 + S_4 = \tfrac{1}{2}\pi\,(\sqrt{2})^2 = \pi.$$

The area of the whole square,

$$S_1 + S_2 + S_3 + S_4 = (2\sqrt{2})^2 = 8$$

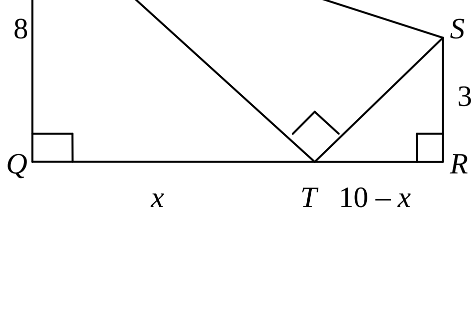
If we now subtract the third equation from the sum of the first two, we get

$$S_1 - S_2 = 3\pi - 8.$$

2. Prove that among any 18 consecutive three-digit numbers there must be one which is divisible by the sum of its digits.

Well, I guess anyone who has been around a bit will know that "being divisible by the sum of its digits" is close to the tests for divisibility by either 3 or 9. Except that it doesn't quite say this. But I suppose the mention of 18 consecutive numbers gives a bit of a clue that we are on the right track. Divisibility by 9 gives more of a chance that the sum will be a factor of the number, so let's try that.
Now in fact, if you have a 3-digit number divisible by 9, the only three possibilities for the sum of the digits are sums of 9 or 18 or 27.
In any set of 18 consecutive numbers, we must have two multiples of 9, one of which is odd and one is even.
If either has a digit sum of 9, then we are done, because that number is divisible by 9.
The only way one can have a sum of 27 is to have the number 999, and this is divisible by 27.
So the only other possibility we have to consider is that both have a digit sum of 18. In this case, both are divisible by 9, and since one of them is even, this will be divisible by 18. This completes the proof.

3. In a trapezium PQRS, PQ and RS are perpendicular to QR, PQ = 8, QR =10 and RS = 3. T is a point on QR such that PTS = 90^o. Find all possible values for the area of ΔPTS.



Since $PTS = 90^o$, $QTP = TSR$, and hence $\Delta QTP \sim \Delta RST$.
If we let $QT = x$, then $TR = 10 - x$, and so

$$x/8 = 3/(10 - x)$$

leading to $x^2 - 10x + 24 = 0$, so $(x - 6)(x - 4) = 0$
and hence $x = 4$ or 6.
The area of $\Delta PTS = \tfrac{1}{2}(8 + 3) \times 10 - 4x - \tfrac{1}{2} \times 3(10 - x)$

$$= 55 - 4x - \tfrac{1}{2} \times 3(10 - x)$$

If $x = 4$, $[PTS] = 55 - 16 - 9 = 30$, whereas
if $x = 6$, $[PTS] = 55 - 24 - 6 = 25$.

4. Find all solution pairs (n, m) of $2^n + 7 = m^2$ in which n and m are both integers. [You must prove there are no other solutions.]

The hint would be to use modular arithmetic (or consider remainders when dividing by something). The question is "mod what?".
We have a square, and squares sometimes work well mod 3, since $0^2 \equiv 0$, $1^2 \equiv 1$, and $2^2 \equiv 1 \pmod 3$. So everything is equivalent either to 0 or 1 (mod 3). i.e. no square can give 2 (mod 3).
They also work well mod 4, since $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 0$ and $3^2 \equiv 1$. So squares only give 0 or 1 (mod 4); they cannot give 2 or 3 (mod 4).
In fact they work quite well mod 8, since $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 4$, $3^2 \equiv 1$, $4^2 \equiv 0$, $5^2 \equiv 1$, $6^2 \equiv 4$, $7^2 \equiv 1$. So all squares are either 0, 1 or 4 (mod 8). No square can give 2, 3, 5, 6 or 7 (mod 8).

In this case arithmetic mod 4 will work well.
We first note that $n < 0$ will not give an integer and $n = 0$ will not work, so $n > 0$, in which case 2^n is even, so m will be odd. So $m^2 \equiv 1 \pmod 4$ and therefore $m^2 - 7 \equiv 2 \pmod 4$.
But if $n > 1$, then 2^n will give 0 (mod 4), so the only possibility is that $n = 1$, and hence $m = \pm 3$.
So we have $(n, m) = (1, 3)$ or $(1, -3)$.

5. In an urn, there are only black and white marbles, the total number of which, rounded to the nearest hundred, is 1000. The probability of pulling out two black marbles is $^{17}/_{43}$ greater than the probability of pulling out two white marbles. How many of each type of marble are there in the urn?

Here is a possible approach – and I suspect the simplest.
Suppose there are b black and w white marbles. Considering probabilities we find that

$$\frac{b}{b+w} \times \frac{b-1}{b+w-1} = \frac{17}{43} + \frac{w}{b+w} \times \frac{w-1}{b+w-1} \qquad (*)$$

Multiplying up gives

$$43b(b-1) = 17(b+w)(b+w-1) + 43w(w-1)$$

giving $$17(b+w)(b+w-1) = 43\{b(b-1) - w(w-1)\} = 43\{b^2 - w^2 - (b-w)\}$$

So $$17(b+w)(b+w-1) = 43(b-w)(b+w-1)$$

Since $b + w \neq 1$, we can cancel to get $17(b+w) = 43(b-w)$ and hence $13b = 30w$.

So the ratio $b{:}w$ is 30:13 and the only two multiples of 43 which round to 1000 are 989 and 1032. Splitting each of these in the ratio 30:13 gives two possible solutions: either 690 black and 299 white, or 720 black and 312 white. Both of these satisfy equation (*), so these are the two solutions.

6. Every member of a given sequence, beginning with the second, is equal to the sum of the preceding one and the sum of its digits. The first member is 1. Is there, among the members of this sequence, a number equal to 123456?

With a question like this, it is perhaps a good idea to think which answer you are going to be able to prove, and work on that in hope!
If the answer is Yes, I have no idea how I would prove this. One might prove that all numbers in the sequence possess a certain property, but that would not necessarily mean that all numbers with this property are in the sequence.

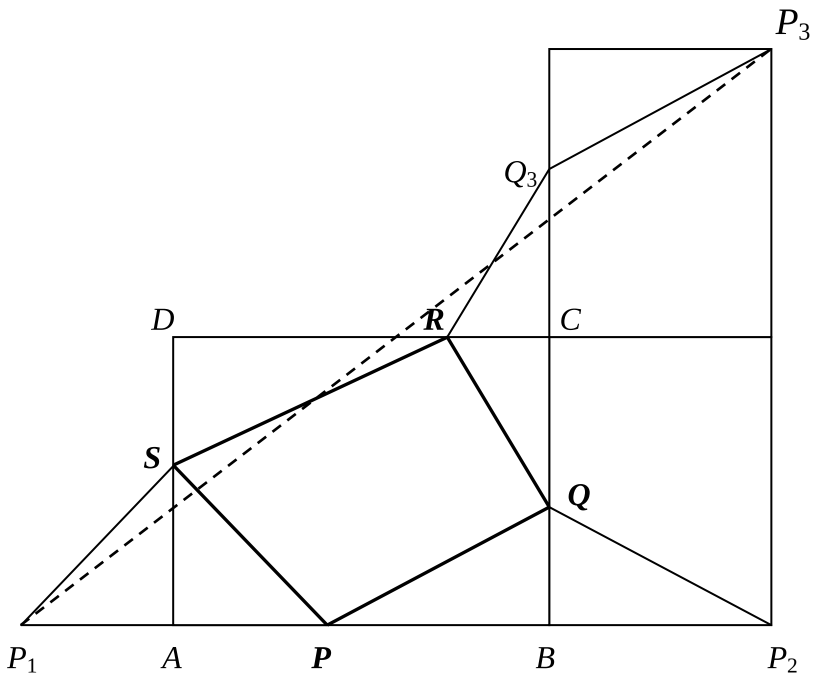
On the other hand, if the answer is No, then if we could find some property possessed by all members of the sequence and show that our number does NOT have this property, then we are done.
So armed with that thought, let's look for a property of all members of the sequence.
The sequence is: 1, 2, 4, 8, 16, 23, 28 (= 23 + 5), 38 (= 28 + 10), 49 (= 38 + 11), etc.
We are not looking for some magic formula (probably), but the point of working out some terms is to begin to get one's head round what is happening.
One hint for mentees might be to think along the lines of Question 2, because this is again about the digit sum of a number and this has to do with divisibility by 3 or 9.
The fact is that mod 3, every number is equivalent to its digit sum (mod 3).
So for example $23 \equiv 2 \pmod 3$ and also $5 \equiv 2 \pmod 3$.
And $28 \equiv 1 \pmod 3$ and also $10 \equiv 1 \pmod 3$.
So if we look at the terms of the sequence (mod 3), we get:

$1,\ 1 + 1 = 2,\ 2 + 2 = 1,\ 1 + 1 = 2$, etc.

The terms alternate between 1 and 2 (mod 3).
But $123456 \equiv 0 \pmod 3$, so does not occur in the sequence.

7. Inside a given rectangle is inscribed a quadrilateral, which has a vertex on each side of the rectangle. Prove that the perimeter of the inscribed quadrilateral is at least double the length of a diagonal of the rectangle.

One could look for an algebraic proof of this, but please encourage mentees to look for a geometric proof, based on the fact that the shortest distance between two points is a straight line, and any other route is at least this length. This needs inspiration and creativity.



In the diagram above, let the original rectangle be *ABCD*, and the inscribed quadrilateral be *PQRS*.
Now reflect *P* in *AD* to get P_1, and also in *BC* to get P_2. Then reflect P_2 and *Q* in *DC* (extended) to get P_3 and Q_3.
First, we see that P_1P_2 is twice *AB*, and P_2P_3 is twice *BC*, so P_1P_3 is twice the diagonal *AC*.
Now we note that $PS = P_1S$, $QR = Q_3R$, and $PQ = P_2Q = P_3Q_3$.
So the sum of the sides of *PQRS* is the length of the zig-zag route $P_1SRQ_3P_3$, which is clearly at least the length of P_1P_3, i.e. at least twice the length of the diagonal.

Equality incidentally would be when S and R are at the points where the dotted line crosses the rectangle, P is where it is, and $PQRS$ is a parallelogram. It might be good to encourage mentees to ask this question after they have done this (or been shown it). Is this parallelogram uniquely optimal? Alternatively, suggest that your mentees look for the equality case right at the start, as this can often suggest a method of proof.

> *8. Theo has four children. The age in years of each child is a positive integer between 2 and 16 inclusive, and all their ages are different. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time, the sum of the ages of the oldest and youngest will be equal to the sum of the squares of the other two. Find all possibilities for their ages.*

This is clearly going to need algebra. It is worth thinking about whether it is best to use letters to denote the ages now or a year ago. If you do it now, then you are going to have to subtract 1 from all the letters for the first equation and add 1 to each of the letters for the second equation. So maybe for simplicity it is good to use letters for their ages 1 year ago.

So suppose their ages a year ago were a, b, c, d where without loss of generality we can assume that $1 \leq a < b < c < d \leq 15$. We also note that $b \leq 13$ so that $b - a \leq 12$.

So we know that $d^2 = a^2 + b^2 + c^2$. (1)
And also that $(d + 2)^2 + (a + 2)^2 = (b + 2)^2 + (c + 2)^2$. (2)
Multiplying out, $d^2 + 4d + 4 + a^2 + 4a + 4 = b^2 + 4b + 4 + c^2 + 4c + 4$
Subtracting gives $4(d + a) + a^2 = 4(b + c) - a^2$,
so $a^2 = 2(b + c - a - d)$ (3)
So a must be even, and since $d > c$, $a^2 = 2(b - a + (c - d)) < 2(b - a) < 24$.
So there are only two possibilities: $a = 2$ or 4.

If $a = 4$, then since $a^2 < 2(b - a)$, we have $2b > a^2 + 2a = 24$, so $b > 12$ and then only possibility is that $b = 13$, $c = 14$ and $d = 15$. This contradicts equations (1) and (2) so there is no solution with $a = 4$.

If $a = 2$, then equation (3) gives $b + c - d = 4$. Substituting $a = 2$ and $d = b + c - 4$ back into (1) and simplifying gives $bc - 4b - 4c + 6 = 0$, leading to $(b - 4)(c - 4) = 10$.
Now $b - 4$ and $c - 4$ are integers between -2 and 10.
There are only two ways to express 10 as a product of two integers in this range: 10×1 and 5×2.
Since $c > b$ and $d = b + c - 4$, we have two possibilities:

$c = 14, b = 5, d = 15$ and $c = 9, b = 6, d = 11$.

Both of these possibilities satisfy equations (1) and (2).
Hence there are just two possibilities for the ages of the children: (3, 6, 15, 16) and (3, 7, 10, 12).

I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to mentoring@ukmt.org is of course also very welcome.