

1. Find all solutions to the sum:
 [Different letters stand for different digits. There may be one or more solutions, but you must show there are no others.]

	TE
	+ TE
	+ <u>TE</u>
	= <u>AT</u>

Since the two-digit number “TE” still has two digits when tripled, it must be less than 34, so $T = 1, 2$ or 3 . We now need to investigate cases:

1. If $T = 1$, then since $3E$ ends in 1, E can only be 7, and $17 + 17 + 17 = 51$, so we have a valid solution.
2. If $T = 2$, then since $3E$ ends in 2, E can only be 4, and $24 + 24 + 24 = 72$, so we have a valid solution.
3. If $T = 3$, then since $3E$ ends in 3, E can only be 1, and $31 + 31 + 31 = 93$, so we have a valid solution.

2. Find the number of positive divisors (factors) of 10^{2013} which are not factors of 10^{2012} .

Firstly we note that the prime factorisations of these two numbers are $2^{2013} \times 5^{2013}$ and $2^{2012} \times 5^{2012}$. So for any number to be a factor of one but not the other must have one of these factors to the power 2013. So either we have $2^{2013} \times 5^n$, where n can take any value from 0 to 2013, i.e. 2014 possibilities, or we have $2^n \times 5^{2013}$, again 2014 possibilities, except that we have counted $2^{2013} \times 5^{2013}$ twice. So the answer is $2014 \times 2 - 1 = 4017$.

3. Solve the system of equations

$x^2 + xy + y^2 = 21$
$x^2 - xy + y^2 = 13$

If we look at the equations, it looks like it is going to involve $(x + y)^2$ and $(x - y)^2$. The equations suggest finding the difference, which gives $2xy = 8$, so $xy = 4$. If we add this to the first equation, we get $(x + y)^2 = 25$, so $x + y = \pm 5$. Similarly subtracting xy from (2) gives $(x - y)^2 = 9$, so $x - y = \pm 3$. Adding these two results gives $2x = \pm 8$ or ± 2 , so $x = \pm 4$ or ± 1 . Since $xy = 4$, we get four ordered pairs for (x, y) of $(4, 1)$, $(-4, -1)$, $(1, 4)$ and $(-1, -4)$. We now need to check that all these fit the original equations, which they do.

4. Four positive integers a, b, c and d have a product of $8!$ and satisfy the equations

$$ab + a + b = 524$$

$$bc + b + c = 146$$

$$cd + c + d = 104.$$

Find a, b, c and d .

First you should have a think about the equations, and what you might do with them. Note that we are told that a, b, c, d are integers, so we are into Number Theory rather than just solving generally. This often suggests factorisation is important, so we should look for ways to do this.

The left hand sides do not factorise, but if we add 1 to each side we can factorise and get

$$(a + 1)(b + 1) = 525 = 3 \cdot 5^2 \cdot 7$$

$$(b + 1)(c + 1) = 147 = 3 \cdot 7^2$$

and $(c + 1)(d + 1) = 105 = 3 \cdot 5 \cdot 7$

So although we do not want factors of $a + 1, b + 1$ etc, it's all we've got, so we might as well see where we can go from here.

We can see from the middle line that neither $b + 1$ nor $c + 1$ can be a multiple of 5, so $d + 1$ must be a multiple of 5, and $a + 1$ must be a multiple of 25. And also $c + 1$ must be a multiple of 7.

Now $a + 1$ is a multiple of 25, and from the above there are only three possibilities for $a + 1$, since $b + 1 > 1$, namely $a + 1 = 25, 75$ or 175 . These give possible values for a of 24, 74 and 174. But we note that since $8! = 2^7 \times 3^2 \times 5 \times 7$, a cannot have any prime factors apart from 2, 3, 5 and 7, so a cannot be $74 = 2 \times 37$ or $174 = 6 \times 29$. So $a + 1 = 25, b + 1 = 21, c + 1 = 7$ and $d + 1 = 15$. This gives $(a, b, c, d) = (24, 20, 6, 14)$ and these numbers do indeed multiply to $8!$

5. $A, B,$ and C stand in a triangle and are each given a card to hold out in front of themselves. Each card has a single, positive, non-zero digit on it which the holder can not see but is visible to the other two. They are told that the values of the cards are such that $A < B < C$ and that the product of the three digits is a multiple of 12. Starting with A , they are asked in turn whether they can state with certainty the value of their own card. If not they must pass. A, B and then C all pass and then A states the value of his card and still in turn, B and C do the same. What digit does each person hold?

I did this by trying the possibilities. Since $A < B < C$ and the product is a multiple of 12, I think I made it 38 possible triples. [The triple must contain either 3, 6 or 9. And either 4 or 8 or both 2 & 6.]

The triples 126, 236, 345, 347, 348, 358, 459, 479 are all impossible because A would have known his number in each case from seeing the other two.

Then the triples 134, 146, 234, 267, 346, 456, 467, 468, 568, 589, 678, 689, 789 will all be eliminated because B would have known his number. [Note: B would have eliminated 146 because he would have known his number couldn't have been 2 because if it had been, A would have known his.]

Then the triples 138, 149, 168, 189, 256, 289, 349, 368, 378, 389, 469, 489 would have been eliminated because C would have known his number from the other two having eliminated the possible ones from the first two rounds.

This only leaves 238, 246, 249, 268, 269.

A thus knows his card must be a "2" so he states it.

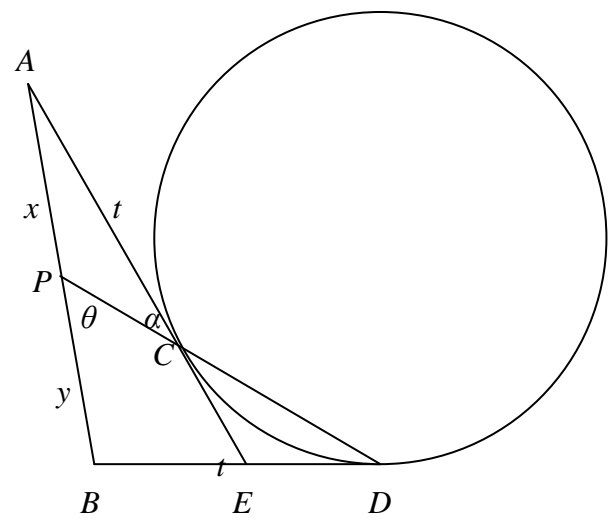
If B knows his card, he cannot see 2 and 8 or 2 and 9, otherwise he wouldn't know, so he must be seeing 2 and 6, and he then states that his card is a "4".

C knows his card cannot have been a 9, otherwise B would not have known his.

So he knows that his card is a "6". Hence the triple (A, B, C) is $(2, 4, 6)$.

6. From any two points C and D on the circumference of a circle, two tangents of equal length are drawn with orientations as shown. Prove that the line DC extended bisects AB.

I tried all sorts of methods on this and in the end hit on the Sine Rule.
 Firstly, if we extend tangent AC to meet BD at E, we note that EC = ED since tangents from a point are equal in length. So ΔECD is isosceles and if CDE = α, then ECD = PCA = α.
 If the tangent length is t, and BPD = θ, and AP = x and BP = y,
 in ΔAPC, $x/\sin \alpha = t/\sin (180-\theta)$
 $= t/\sin \theta$
 and in ΔBPD, $y/\sin \alpha = t/\sin \theta$
 Therefore x = y, and BC bisects AB.



7. If the digits "15" are inserted in the middle of the square 16, the number 1156 is formed, which is also a perfect square. If this process is repeated, we obtain the numbers 111556, 11115556, 1111155556 and so on. Prove that each number in this sequence is a square.

I found this amazingly difficult and tried all sorts of things including proof by induction (horrible!), and lots more.

Firstly a little bit of arithmetic tells us that $1156 = 34^2$ and $111556 = 334^2$ etc, so we conjecture that these numbers are the squares of the numbers $333...34^2$ for different numbers of 3's. The problem is to prove that this continues to work.

I tried writing this as $(33...3 + 1)^2$, but this did not seem to help.

Here is the method I have come up with.

Firstly, note that the square of any number is 1 more than the product of the two numbers either side of it. e.g. $8^2 = 7 \times 9 + 1$. The reason for this is that $(n - 1)(n + 1) = n^2 - 1$.

So $33...34^2$ (with n 3's) = $33...33 \times 33...35 + 1$

So it is sufficient to prove that

$$\underset{(n+1 \text{ 3's})}{33...33} \times \underset{(n \text{ 3's})}{33...35} = \underset{(n \text{ 1's, } n \text{ 5's})}{11...155...5}$$

Note that $3 \times \underset{(n \text{ 3's})}{33...35} = 3 \times \underset{(n+1 \text{ 9's})}{33...33} + 3 \times 2 = \underset{(n \text{ 0's})}{99...99} + 6 = 100...005$

So $\underset{(n+1 \text{ 3's})}{33...33} \times \underset{(n \text{ 3's})}{33...35} = \underset{(n+1 \text{ 1's})}{11...11} \times 3 \times \underset{(n \text{ 3's})}{33...35} = \underset{(n+1 \text{ 1's})}{11...11} \times \underset{(n \text{ 0's})}{100...005}$

which is $\underset{(n \text{ 1's, } n \text{ 5's})}{11...155...5}$ as required.

This proof is better written out using Sigma Notation, for those who are familiar with this.

8. Ten girls, numbered from 1 to 10, sit at a round table, in a random order. Each girl then receives a new number, namely the sum of her own number and those of her two neighbours. Prove that some girl receives a new number greater than 17.

Solution 1

Proof by contradiction (we suppose the opposite to be true and show that it leads to a contradiction).

The sum of the numbers is $1 + 2 + \dots + 10 = 55$; each number contributes to three of the new numbers, so the total of the new numbers is $3 \times 55 = 165$. Assume that none of the new numbers is greater than 17. If more than five of the new numbers are equal to 17, then there are two neighbours with the same new numbers; suppose they sit on seats x and $x + 1$. This implies that the girls in position $x - 1$ and $x + 2$ (possibly taken modulo 10) had the same old number, which is impossible.

Therefore, there are not more than five girls whose new number is 17, and the sum of all the new numbers is at most $5 \times 16 + 5 \times 17 = 5 \times 33 = 165$. Since equality holds, there are precisely five girls whose new number is 16, and five girls whose new number is 17. By the above argument, the numbers of girls who sit next to each other must be distinct, and so they must form a 16—17—16. . . pattern. Consider two girls on seats x and $x + 1$ again. Their new numbers are 16 and 17, which shows that the old numbers of the girls in position $x - 1$ and $x + 2$ differ by exactly 1 (and this holds for any x !). If y is the position of the girl whose old number was 1, then this argument shows that the girls in positions $y - 3$ and $y + 3$ are either 0 (which is impossible) or 2. But there was only one girl who got number 2, and so we finally arrive at a contradiction.

Solution 2

Divide the nine girls with numbers 2 to 10 into three groups of three: the three girls to the left of no 1, the three girls to the right of no 1, and the three girls opposite no 1. The total sum of their numbers is 54, so the sum of at least one group must be at least 18. Therefore the new number of the girl in the middle of that group is at least 18.