BMOS MENTORING SCHEME (Intermediate Level)
February 2013 (Sheet 5)
Hints and Solutions for Mentors and Teachers

Here are comments and solutions for Sheet 5.

> 1. *Find the smallest positive integer that appears in all the arithmetic sequences:*
> *22, 33, 44, 55, ...*
> *24, 37, 50, 63, ...*
> *25, 39, 53, 67, ...*

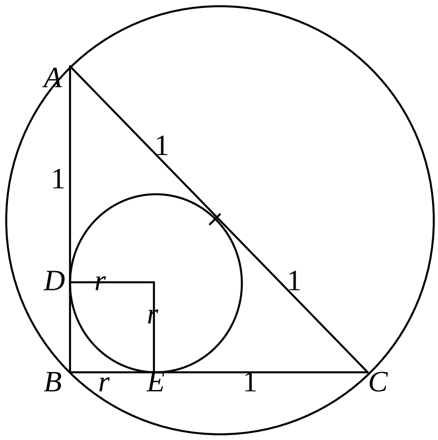
A quick look at the sequences tells us that the common differences are 11, 13 and 14. So if there is a common number in the sequences, then another one will occur after a gap of $11 \times 13 \times 14 = 2002$. If we can find a common element, then we can find another. One suspects a Machiavellian mind of making up such a problem, and indeed this is the case because if you go back one step in each sequence, you see that the number before the first one given in each case is 11. So the next common number will be 11 + 2002 which is 2013. (It had to come in somewhere!!)

> 2. *Find the number of digits in $5^{20} \times 4^{13}$ when it is worked out and written as a single number.*

We can write 4^{13} as 2^{26}, and $5^{20} \times 2^{26} = 10^{20} \times 2^6 = 64 \times 10^{20}$, so the number has 22 digits.

> 3. *An isosceles right-angled triangle is inscribed inside a circle of radius 1. Find the radius of the incircle of the triangle.*

Since the triangle is right-angled, the hypotenuse is a diameter of the circle. And using the fact that tangents to a circle from a point are equal in length, we can see that $AD = EC = 1$ on my diagram.
If the radius of the incircle is r, then $AB = BC = r + 1$.
So, by Pythagoras, $2.(r + 1)^2 = 4$
Therefore $(r + 1)^2 = 2$,
and hence $r + 1 = \pm\sqrt{2}$
so $r = -1 \pm \sqrt{2}$
But, $r > 0$, so $r = -1 + \sqrt{2}$.



> 4. *If $x + y = 5$ and $xy = 2$, find the values of $x^3 + y^3$ and $x^4 + y^4$.*

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x + y)$.
So $5^3 = x^3 + y^3 + 3 \times 2 \times 5$, and hence $x^3 + y^3 = 125 - 30 = 95$.
We are going to need $x^2 + y^2$ for the next bit, and $x^2 + y^2 = (x + y)^2 - 2xy = 25 - 4 = 21$.
Now $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^4 + y^4 + 4xy(x^2 + y^2) + 6x^2y^2$.
So $5^4 = x^4 + y^4 + 4 \times 2 \times 21 + 6 \times 2^2$, and hence $x^4 + y^4 = 625 - 168 - 24 = 433$.

5. Four dice, coloured red, blue, green and black are rolled. In how many ways can the product of the numbers rolled be 36 (where, for example, a red 4 is considered different from a blue 4)?

First we need to identify the combinations which have a product of 36. Mentees need to have a systematic way of doing this so they can be sure all have been accounted for. e.g. considering the numbers of 6's:
Two sixes: 6, 6, 1, 1 only. One six: 6, 3,2,1 only No sixes: 3, 3, 2, 2 or 3, 3, 4, 1.
Now we need to multiply by the number of orderings ("permutations") of each:
6611 can be done in 6 ways (either list them or use ${}^4C_2$ = 4!/(2!2!) = 6, choosing two places for the 6's)
6321 can be done in 4! = 24 ways.
3322 can be done in ${}^4C_2$ = 6 ways as above
3341 can be done in ${}^4C_2 \times 2$ = 12 ways, since the 41 can be done either way for any pair of 3's.
So the total number of ways is 6 + 24 + 6 + 12 = 48.

6. From the set of consecutive integers {1, 2, 3, ... n}, five integers that form an arithmetic sequence are deleted. The sum of the remaining integers is 5000. Determine all values of n for which this is possible and for each acceptable value of n, determine the number of five-integer sequences possible.

This first appears to have not much information, but of course if n is large, then removing just five numbers will leave totals of much more than 5000. So we might start by trying to put some bounds on n. It will of course be important for mentees to be familiar with the fact that the sum of the first n integers 1 + 2 + ... + n = ½n(n + 1), the n^{th} Triangle Number.
So we must have ½n(n + 1) > 5000, and hence n (n + 1) > 10000, and since 99 × 100 = 9900, we must have $n \geq 100$.
But also n must not be too much, because even if we remove the highest five numbers, the total is in danger of being more than 5000. So in fact if we add the numbers up to $n - 5$, we must not get more than 5000.
It also helps to note that the sum of five integers in an arithmetic sequence must be 5 × the middle number, i.e. a multiple of 5, so since 5000 is a multiple of 5, we must have ½n(n + 1) as a multiple of 5, i.e. n(n + 1) must be a multiple of 5. This will only happen if n is either a multiple of 5, or 1 less. We can say $n \equiv 0$ or -1 (mod 5). So since $n \geq 100$, we need only consider n = 100, 104, 105, 109, 110 etc. But if n = 105, ½n(n + 1) = 5565, so we need to remove 5 numbers adding to 565. This would need an average of 113, which is impossible. So the only possible values to consider are 100 and 104.
If n = 100, ½n(n + 1) = 5050, and we need to remove five numbers whose average is 10.
This can be done in four ways: 8, 9, 10, 11, 12 or 6, 8, 10, 12, 14 or 4, 7, 10, 13, 16 or 2, 6, 10, 14, 18.
If n = 104, ½n(n + 1) = 5460, and we need to remove five numbers whose average is 460/5 = 92.
The top three numbers can either be 92, 93, 94 or 92, 94, 96 or 92, 95, 98 or 92, 96, 100 or 92, 97, 102, or 92, 98, 104, i.e six ways.

7. Determine (without a calculator!) the value of $\dfrac{1^4 + 2012^4 + 2013^4}{1^2 + 2012^2 + 2013^2}$.

Firstly, anyone who is going to work out 2013^4 etc longhand, and then divide by some horrendous number deserves both by respect and my sympathy (but more of the latter!).

When one starts to think about this problem, it is a reasonable conjecture that the specific numbers 2012 and 2013 are not important, but that equally well one might have 2010 and 2011 or whatever.

So a good idea is to use algebra and see what we can find out about $1 + n^4 + (n + 1)^4$ and also $1 + n^2 + (n + 1)^2$.

Now, $1 + n^2 + (n + 1)^2 = 1 + n^2 + n^2 + 2n + 1 = 2(1 + n + n^2)$,
and $1 + n^4 + (n + 1)^4 = 1 + n^4 + n^4 + 4n^3 + 6n^2 + 4n + 1 = 2(1 + 2n + 3n^2 + 2n^3 + n^4)$
and a little experimenting will lead to the fact that this is also $2(1 + n + n^2)^2$.

So $\dfrac{1+n^4+(n+1)^4}{1+n^2+(n+1)^2} = \dfrac{2(1+n+n^2)^2}{2(1+n+n^2)} = 1+n+n^2$.
And hence if $n = 2012$, the value is $1 + 2012 + 2012^2$, which, with a little long multiplication, turns out to be 4050157.

> *8. Determine the least natural number n for which the following result holds:*
> *No matter how the elements of the set {1, 2, ..., n} are coloured red or blue, there are integers x, y, z, w in the set (not necessarily distinct) of the same colour such that* $x + y + z = w$.

Firstly, this old BMO1 question needs some thinking about! It is not saying that for some n, there is a colouring for which this equation holds. It is saying that no matter how the elements are coloured, this is bound to happen for some set of four elements (which could be the same or different). So maybe it is helpful to think about the opposite. i.e. Find the largest possible number which can be done without this occurring.

So this needs building up slowly (at least it does for me!). The number "1" has to be some colour. Suppose without loss of generality that this is Red. Then to avoid having the condition, since 1 + 1 + 1 = 3, we must have 3 as Blue. But we cannot say anything about any other small numbers. Let's consider the number 2.

Case 1: If 2 is Red, then since 1 + 1 + 2 = 4, 4 must be Blue. For similar reasons, so must 5 and 6. It does not really matter what 7 and 8 are since they cannot be made up by three elements of the same colour. But 9 = 3 + 3 + 3 must be Red and so must 10 = 3 + 3 + 4.

We now have:	**RED:**	**1, 2, 9, 10**	**Doesn't matter: 7, 8**
	BLUE:	**3, 4, 5, 6**	

But we have a problem with 11, since 1 + 1 + 9 = 11, so it can't be Red, but also 3 + 3 + 5 = 11, so it can't be Blue. So we can avoid the condition if $n = 10$ (with say **Red 1, 2, 7, 8, 9, 10 Blue 3, 4, 5, 6**), but we cannot accommodate 11 if 1 and 2 are both Red. Now we consider Case 2:

Case 2: If 1 is Red but 2 is Blue, then again 3 must be Blue since 1 + 1 + 1 = 3. It does not matter what 4 and 5 are, but 6 = 2 + 2 + 2 must be Red, and similarly 7 = 2 + 2 + 3 and 8 = 2 + 3 + 3.

We now have:	**RED:**	**1, 6, 7, 8**	**Doesn't matter: 4, 5**
	BLUE:	**2, 3**	

But we now have a problem with 9, since 9 = 1 + 1 + 7, so it can't be Red.
And also 9 = 3 + 3 + 3 so it can't be Blue. So in the Case 2 scenario, we can't accommodate 9.

Conclusion: We can accommodate the numbers 1 to 10 as shown in the example above, but in neither case can we accommodate 11. So the smallest n for which we are **bound** to get a monochromatic quadruple (x, y, z, w) such that $x = y + z = w$ is 11, no matter how the numbers are coloured.

I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to mentoring@ukmt.org is of course also very welcome.