BMOS MENTORING SCHEME (Intermediate Level)
January 2013 (Sheet 4)
Comments and Solutions

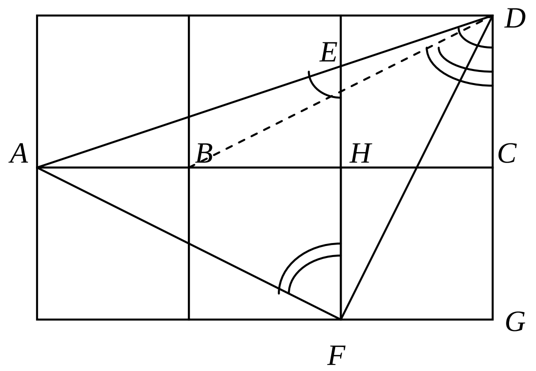
Here are the solutions for Sheet 4. Some nice challenging problems here – lots of algebra!

> *1. A sequence has first term 2013, after which every term is the sum of the squares of the digits of the preceding term. So the second term is $2^2 + 0^2 + 1^2 + 3^2 = 14$, the third term $1^2 + 4^2 = 17$, and so on. Find the 2013th term of the sequence.*

Well the only hope is that some sequence becomes apparent, so you should work out terms until something happens... We have $u_2 = 14$, $u_3 = 17$. Continuing gives 50, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58 and we now see that the next term $u_{16} = 89 = u_8$ so the sequence will continue with period 8 and every $u_{8k} = 89$. Since 2013 = 251 × 8 + 5, the 2013th term is 16.

> 2. There are three squares in the picture. Find the sum of angles *ADC* and *BDC*.

Although this question can be done approximately using Trig, this is NOT acceptable, because it is not exact.



We need to find some clever way of dealing with this. Here is one method:

We draw a second set of three squares, and note that triangles *BCD* and *AHF* are congruent. So angle *BDC* = *AFH*.
We also note that angle *ADC* = *AEF* (alternate angles).
So *ADC* + *BDC* = *AEF* + *AFE* = 180° – *EAF*.
To find *EAF*, we note that ΔADF is isosceles as *AF* = *DF* and we also note that angle *AFD* = 90° since ΔDGF rotates through 90° about *F* to give ΔAHF. So angle *DAF* = 45°; hence *ADC* + *BDC* = 135°.

> 3. *Define the "reverse" of a number as the number obtained by writing its digits in the opposite order, e.g. the reverse of 379 is 973. Find all three-digit numbers with the following property: If we divide the number by its reverse, we get a quotient of 3 and a remainder of the sum of its digits.*

We are looking for a number "*abc*" such that "*cba*" = "*abc*"× 3 + *a* + *b* + *c*.
Using the fact the number "*abc*" = $100a + 10b + c$ and similarly for "*cba*", we have
$100c + 10b + a = 3(100a + 10b + c) + a + b + c$, giving $32a = 100c + 7b$.
Since $1 \le a \le 9$, we deduce that $1 \le c \le 3$, and we now have three cases.
1) If $c = 1$, then $100 \le 32a = 100 + 7b \le 163$, which implies $4 \le a \le 5$. If $a = 4$, then $128 = 100 + 7b$ and $b = 4$. If $a = 5$, then $160 = 100 + 7b$ and b is not an integer.
2) If $c = 2$, then $200 \le 32a = 200 + 7b \le 263$, which implies $7 \le a \le 8$. If $a = 7$, then $224 = 200 + 7b$ and b is not an integer. If $a = 8$, then $256 = 200 + 7b$, giving $b = 8$.
3) If $c = 3$, then $300 \le 32a = 200 + 7b \le 363$, which implies $a \ge 10$, a contradiction.
So the possible solutions are 441 and 882.

4. A convex polygon with n sides has exactly three obtuse angles. Find all possible values of n.

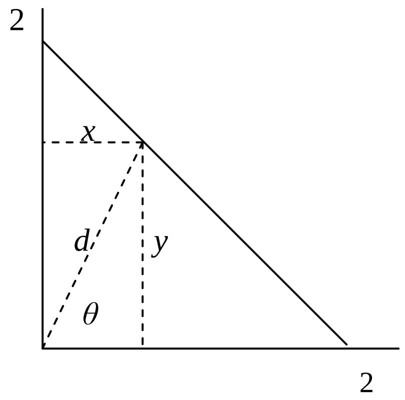
We know that the sum of the angles of an n-gon is $(n - 2) \times 180^o$.
Since exactly three angles are obtuse, we know that these three lie (strictly) between 90^o and 180^o, whereas the other $n - 3$ angles lie (strictly) between 0 and 90^o.

So we have $(n - 3) \times 0 + 3 \times 90 < (n - 2) \times 180 < (n - 3) \times 90 + 3 \times 180$.
Hence $3 < 2n - 4 < n + 3$.
So we must have $2n > 7$, i.e. $n > 3½$ and also $n < 7$.
So the only possible values of n are 4, 5 or 6.
It just remains to show that these are all possible, and this can be easily done.

5. Let x and y be positive real numbers such that $x + y = 2$. Prove that $x^2y^2(x^2 + y^2) \le 2$.

This interesting little problem suggests possibly the use of the AM-GM inequality, which is worth restating here: "For any positive quantities, $a_i,\ldots,a_n$, $\dfrac{a_1 + \ldots + a_n}{n} \ge \sqrt[n]{a_1 \ldots a_n}$." i.e. the Arithmetic Mean ≥ Geometric Mean. So we can see that $\sqrt{(xy)} \le (x + y)/2 = 1$. So $xy \le 1$, but the problem is that we have $x^2 + y^2$ to deal with and this is more like an AM, so the inequality is the wrong way. It is often worth seeing when equality holds, and as one might expect, it holds here when $x = y = 1$. It is tempting to use the fact that $x^2 + y^2 = (x + y)^2 - 2xy$, but again this leads to an inequality the wrong way round. Another idea might be to think graphically. Since $x + y = 2$, and $x, y > 0$, the point (x, y) lies internally on the line joining (2, 0) and (0, 2). But it is not clear where to go from there. We can see that the distance of (x, y) from O is $\sqrt{(x^2 + y^2)}$ and this lies between $\sqrt{2}$ at (1, 1) and 2 at the ends.

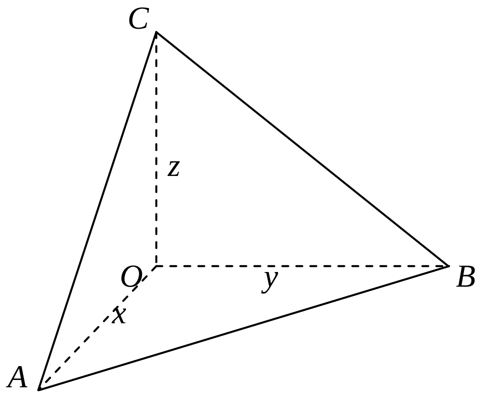
So how about this for an argument:
We are seeking effectively to maximise the product of the area of the rectangle from the origin to (x, y) multiplied by the length of its diagonal.
If we call the diagonal length d, then $x = d.\cos\theta$, and $y = d.\sin\theta$, so we are seeking to maximise $d^3\sin\theta.\cos\theta$, which equals ½ $d^3\sin 2\theta$, which is maximised when $\theta = 45^o$, i.e. at (1, 1) as required.
Good result, but unfortunately the reasoning is totally spurious since d is variable!
Back to the drawing board!



If anyone can fill in these proofs, either using AM-GM or trig or something geometric, I would be pleased to know!
For now, all I can give is a straight algebraic proof:
Since $x + y = 2$, let $x = 1 - s$ and $y = 1 + s$, where $0 \le s < 1$.

Then
$$\begin{aligned} x^2y^2(x^2 + y^2) &= (1 - s)^2(1 + s)^2\{(1 - s)^2 + (1 + s)^2\} \\ &= (1 - s^2)^2(2 + 2s^2) \\ &= 2(1 - s^2)(1 - s^4) \\ &\le 2. \end{aligned}$$

6. *In the tetrahedron shown in the diagram, angles COA, AOB, COB are right angles. The three triangles meeting at O have areas of 6, √39 and 5 units. Determine the area of ΔABC.*



If we let the lengths *OA*, *OB*, *OC* be *x*, *y*, *z*, then since the areas are 6, √39 and 5, we have $xy = 12$, $yz = 2\sqrt{39}$ and $zx = 10$.

Now I don't have a clever way to do this. I would be very interested to hear of one. The area of Δ*ABC* is ½*ab.sin C* where $a = \sqrt{(y^2 + z^2)}$ and $b = \sqrt{(x^2 + z^2)}$ and by the Cosine Rule,

$$\cos C = \frac{x^2 + z^2 + y^2 + z^2 - x^2 - y^2}{2\sqrt{y^2 + z^2}\sqrt{x^2 + z^2}}$$

$$= \frac{z^2}{\sqrt{y^2 + z^2}\sqrt{x^2 + z^2}}$$

so since $\sin C = \sqrt{(1 - \cos^2 C)}$, $\sin C = \sqrt{1 - \frac{z^4}{(y^2 + z^2)(x^2 + z^2)}} = \sqrt{\frac{y^2x^2 + x^2z^2 + y^2z^2}{(y^2 + z^2)(x^2 + z^2)}}$

so the area of Δ*ABC* is $\frac{1}{2}\sqrt{(x^2 + z^2)}\sqrt{(y^2 + z^2)}\sqrt{\frac{y^2x^2 + x^2z^2 + y^2z^2}{(y^2 + z^2)(x^2 + z^2)}} = \frac{1}{2}\sqrt{y^2x^2 + x^2z^2 + y^2z^2}$

So area = $\frac{1}{2}\sqrt{(12^2 + 4 \times 39 + 10^2)} = \frac{1}{2}\sqrt{(144 + 156 + 100)} = \frac{1}{2} \times 20 = 10$.

7. *Prove that, for any integer a > 1, there is a prime p such that $1 + a + a^2 + ... + a^{p-1}$ is composite.*

If $a > 2$, then $a - 1 > 1$ and there exists a prime *p* that divides $a - 1$. Hence $a \equiv 1 \pmod p$ and $1 + a + a^2 + ... + a^{p-1} = M_p$ (say) $\equiv 1 + 1 + ... + 1 \equiv 0 \pmod p$, and hence is divisible by *p*. Since $M_p \geq 1 + a > p$, M_p must be composite.

Now consider the case $a = 2$. We need to find a value of *p* which works. We do need to go a little way to find this as $p = 3$ gives 7, $p = 5$ gives 31 and $p = 7$ gives 127, all of which are prime. However the next one works as $p = 11$ gives 2047 which is 23×89.

8. i) If $a + b = 3$ and $a^2 + b^2 = 5$, find $a^3 + b^3$.
ii) If $a + b + c = 3$, $a^2 + b^2 + c^2 = 5$ and $a^3 + b^3 + c^3 = 7$, find $a^4 + b^4 + c^4$ and $a^5 + b^5 + c^5$.

i) The point of the first part is simply to give a clue as to how to do the second.
Firstly, $(a + b)^2 = a^2 + b^2 + 2ab$, so $2ab = 3^2 - 5 = 4$, so $ab = 2$.
Now $(a + b)(a^2 + b^2) = a^3 + a^2b + ab^2 + b^3 = a^3 + b^3 - ab(a + b)$
So $3 \times 5 = a^3 + b^3 - 2 \times 3$, so $a^3 + b^3 = 21$.

ii) Using similar ideas, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, so $ab + bc + ca = (3^2 - 5)/2 = 2$.
And $(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + a^2b + b^2a + b^2c + c^2b + c^2a + a^2c$ .... (1)
whereas $(a + b + c)(ab + bc + ca) = a^2b + b^2a + b^2c + c^2b + c^2a + a^2c + 3abc$ .... (2).
From (2), we have $a^2b + b^2a + b^2c + c^2b + c^2a + a^2c = (a + b + c)(ab + bc + ca) - 3abc$.
Substituting in (1) gives: $(a + b + c)(a^2 + b^2 + c^2) = a^3 + b^3 + c^3 + (a + b + c)(ab + bc + ca) - 3abc$.
This gives the useful factorisation (which is worth remembering!):

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

So $7 - 3abc = 3 \times (5 - 2)$, and hence $abc = -\frac{2}{3}$.

Now it's just a question of messing around with expansions to find the things we want.
Since $(a^2 + b^2 + c^2)^2 = 25$,
we have $a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2) = 25$.
But $(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2(a^2bc + ab^2c + abc^2)$,
so $a^2b^2 + b^2c^2 + c^2a^2 = 2^2 - 2abc(a + b + c) = 4 - 2 \times -\frac{2}{3} \times 3 = 8$,
so $a^4 + b^4 + c^4 = 25 - 2 \times 8 = 9$.

To get $a^5 + b^5 + c^5$, note that
$(a^3 + b^3 + c^3)(a^2 + b^2 + c^2) = a^5 + b^5 + c^5 + a^3b^2 + b^3c^2 + c^3a^2 + a^2b^3 + b^2c^3 + c^2a^3$
and $a^3b^2 + b^3c^2 + c^3a^2 + a^2b^3 + b^2c^3 + c^2a^3 = (a^2b^2 + b^2c^2 + c^2a^2)(a + b + c) - (a^2b^2c + ab^2c^2 + a^2bc^2)$
$= (a^2b^2 + b^2c^2 + c^2a^2)(a + b + c) - abc(ab + bc + ca)$
$= 8 \times 3 - -\frac{2}{3} \times 2$
$= 25\frac{1}{3}$
So $a^5 + b^5 + c^5 = 7 \times 5 - 25\frac{1}{3} = 9\frac{2}{3}$