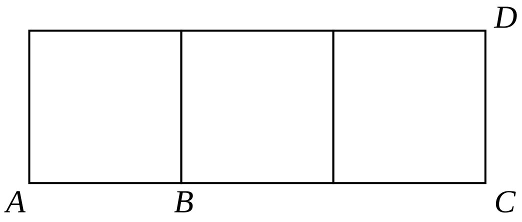
UKMT MENTORING SCHEME (Intermediate Level)
January 2013 (Sheet 4)
Questions

Happy New Year!. Here are some good challenging problems to start 2013.

1. A sequence has first term 2013, after which every term is the sum of the squares of the digits of the preceding term. So the second term is $2^2 + 0^2 + 1^2 + 3^2 = 14$, the third term $1^2 + 4^2 = 17$, and so on. Find the 2013th term of the sequence.

2. There are three squares in the picture.
Find the sum of angles *ADC* and *BDC*.
[*Reminder: no calculators!*]



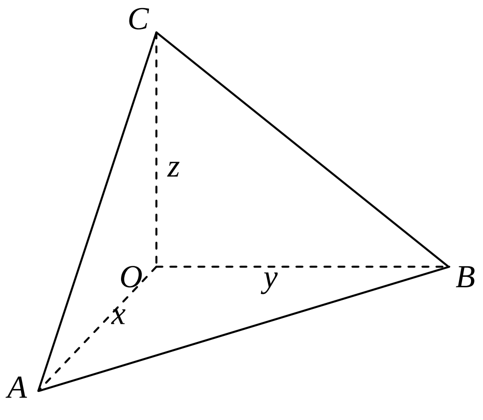
3. Define the "*reverse*" of a number as the number obtained by writing its digits in the opposite order, e.g. the reverse of 379 is 973. Find all three-digit numbers with the following property: If we divide the number by its reverse, we get a quotient of 3 and a remainder of the sum of its digits.

4. A convex polygon with *n* sides has exactly three obtuse angles. Find all possible values of *n*.
[*A "convex" polygon is one where all the interior angles are less than 180°, i.e. if you stretch an elastic band round the polygon, it will go along all the sides.*]

5. Let x and y be positive real numbers such that $x + y = 2$. Prove that $x^2y^2(x^2 + y^2) \leq 2$.

6. In the tetrahedron shown in the diagram, angles *COA*, *AOB*, *COB* are right angles. The three triangles meeting at *O* have have areas of 6, $\sqrt{39}$ and 5 units.

Determine the area of ΔABC.



7. Prove that, for any integer $a > 1$, there is a prime p such that $1 + a + a^2 + ... + a^{p-1}$ is composite.

8. i) If $a + b = 3$ and $a^2 + b^2 = 5$, find $a^3 + b^3$.
ii) If $a + b + c = 3$, $a^2 + b^2 + c^2 = 5$ and $a^3 + b^3 + c^3 = 7$, find $a^4 + b^4 + c^4$ and $a^5 + b^5 + c^5$.

Deadline for receipt of solutions: 31st January 2013