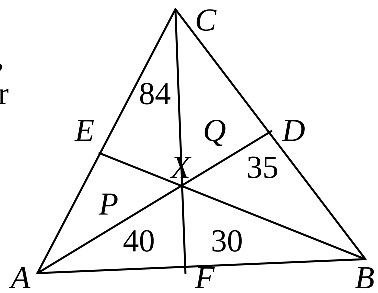
BMOS MENTORING SCHEME (Intermediate Level)
December 2012 (Sheet 3)
Solutions and Comments

Here are the solutions for Sheet 3.

1. As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



If we call the missing areas P and Q, and the points D, E, F, X as shown, we see that $[ABD]:[XBD] = AD : XD$ since these are in proportion to their heights. This ratio is 105 : 35 = 3 : 1. So $AX = 2\,XD$.
Also $[ACD] : [XCD]$ is the same ratio, so we have

$$P + Q + 84 = 3Q \qquad \text{so} \qquad P + 84 = 2Q.$$

Now repeating the same procedure for Δs with base AF and AB,

$$[ACF] : [AXF] = [FCB] : [FXB]$$

So $(Q + 65) / 30 = (P + 124) / 40$,
giving $4Q - 3P = 112$. Solving simultaneously gives $P = 56$ and $Q = 70$.
So the area of ΔABC is $84 + 56 + 40 + 30 + 35 + 70 = 315$.

2. *For x, y, z > 0, satisfying* $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$, *find the value of xyz.*

Solution 1
Multiplying all three expressions together,

$$\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right) = xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz}$$

$$(4)(1)\left(\frac{7}{3}\right) = 4 + 1 + \frac{7}{3} + xyz + \frac{1}{xyz}$$

$$2 = xyz + \frac{1}{xyz}$$

$$0 = (xyz - 1)^2$$

Thus $xyz = 1$.

Solution 2
We have a system of three equations and three variables, so we can apply repeated substitution.

$$4 = x + \frac{1}{y} = x + \frac{1}{1 - \frac{1}{z}} = x + \frac{1}{1 - \frac{1}{7/3 - 1/x}} = x + \frac{7x - 3}{4x - 3}$$

Multiplying out the denominator and simplification yields

$4(4x-3) = x(4x-3) + 7x - 3 \Longrightarrow (2x-3)^2 = 0$, so $x = \frac{3}{2}$. Substituting leads to $y = \frac{2}{5}, z = \frac{5}{3}$, and the product of these three variables is 1.

3. *The expression* $\binom{n}{r}$ means $\frac{n!}{r!(n-r)!}$ *where* $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$.   *Find the largest two-digit prime number which is a factor of* $\binom{200}{100}$.

Expanding the binomial coefficient, we get $\binom{200}{100} = \frac{200!}{100!100!}$. If the prime is p, then $10 \le p < 100$.

If $p > 50$, then the factor of p appears twice in the denominator. Thus, we need p to appear as a factor three times in the numerator, so $3p < 200$. The largest such prime is 61.

4. *The numbers 1447, 1005, and 1231 have something in common: each is a four-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?*

Suppose the two identical digits are both one. Since the thousands digits must be one, the other one can be in only one of three digits, $11xy$, $1x1y$, $1xy1$.
Because the number must have exactly two identical digits, $x \neq y$, $x \neq 1$ and $y \neq 1$.
Hence, there are $3 \times 9 \times 8 = 216$ numbers of this form.
Suppose the two identical digits are not one. Therefore, consider the following possibilities,

$$1xxy, \quad 1xyx, \quad 1yxx.$$

Again, $x \neq y$, $x \neq 1$ and $y \neq 1$. There are $3 \times 9 \times 8 = 216$ numbers of this form as well.
Thus, the desired answer is $216 + 216 = 432$.

5. *Find four positive integers a, b, c and d which have a product of 8! and satisfy* $ab + a + b = 524$ $bc + b + c = 146$ $cd + c + d = 104$

We can rewrite the three equations as follows:

$$(a+1)(b+1) = 525$$
$$(b+1)(c+1) = 147$$
$$(c+1)(d+1) = 105$$

Let $(e, f, g, h) = (a+1, b+1, c+1, d+1)$. We get:

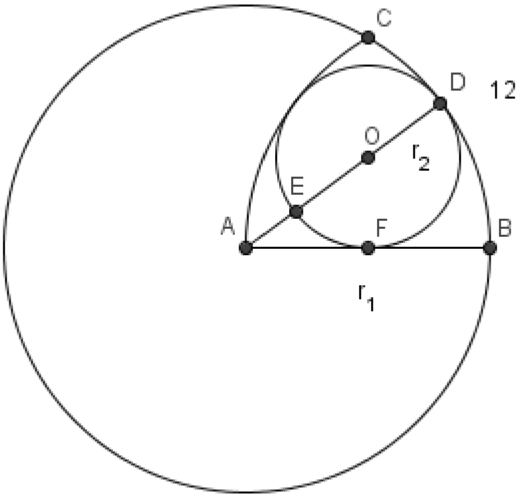
$$ef = 3 \cdot 5 \cdot 5 \cdot 7$$
$$fg = 3 \cdot 7 \cdot 7$$
$$gh = 3 \cdot 5 \cdot 7$$

Clearly 7^2 divides fg. On the other hand, 7^2 can not divide f, as it then would divide ef. Similarly, 7^2 can not divide g. Hence 7 divides both f and g. This leaves us with only two cases: $(f, g) = (7, 21)$ and $(f, g) = (21, 7)$.

The first case solves to $(e, f, g, h) = (75, 7, 21, 5)$, which gives us $(a, b, c, d) = (74, 6, 20, 4)$, but then $abcd \neq 8!$. (We do not need to multiply, it is enough to note e.g. that the left hand side is not divisible by 7.)

The second case solves to $(e, f, g, h) = (25, 21, 7, 15)$, which gives us a valid quadruple $(a, b, c, d) = (24, 20, 6, 14)$.

6. *The circular arcs AC and BC, as shown, have centres at B and A respectively. If the length of the arc BC i s 12, find the circumference of the circle which is tangential to the line AB and also to the arcs AC and BC.*



Since AB, BC, AC are all radii, it follows that $\triangle ABC$ is an equilateral triangle.

Draw the circle with center A and radius AB. Then let D be the point of tangency of the two circles, and E be the intersection of the smaller circle and AD. Let F be the intersection of the smaller circle and $\overline{AB}$. Also define the radii $r_1 = AB, r_2 = \frac{DE}{2}$ (note that DE is a diameter of the smaller circle, as D is the point of tangency of both circles, the radii of a circle is perpendicular to the tangent, hence the two centers of the circle are collinear with each other and D).

By the Power of a Point Theorem (or using similar triangles),

$$AF^2 = AE \cdot AD \Longrightarrow \left(\frac{r_1}{2}\right)^2 = (AD - 2r_2) \cdot AD.$$

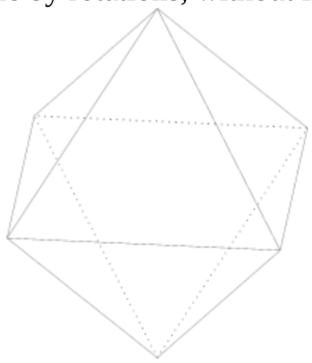
Since $AD = r_1$, then $\frac{r_1^2}{4} = r_1(r_1 - 2r_2) \Longrightarrow r_2 = \frac{3r_1}{8}$. ABC is equilateral, so $\angle BAC = 60^\circ$, and $\overset{\frown}{BC} = 12 = \frac{60}{360} 2\pi r_1 \Longrightarrow r_1 = \frac{36}{\pi}$. Thus $r_2 = \frac{27}{2\pi}$ and the circumference of the circle is 27.

7. *Eight congruent equilateral triangles, each of a different colour, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two coloured octahedra are distinguishable if neither can be rotated to look like the other.)*

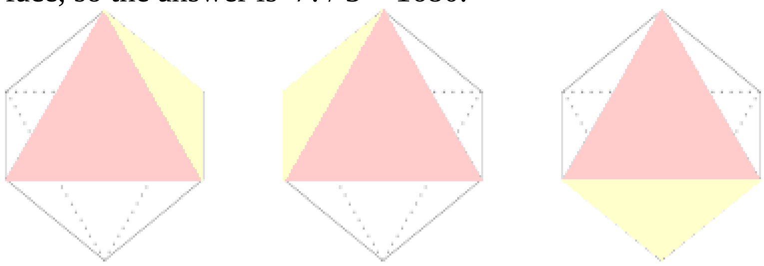
We consider the dual of the octahedron, the cube; a cube can be inscribed in an octahedron with each of its vertices at a face of the octahedron. So the problem is equivalent to finding the number of ways to colour the vertices of a cube.

Select any vertex and call it *A*; there are 8 colour choices for this vertex, but this vertex can be rotated to any of 8 locations. After fixing *A*, we pick another vertex *B* adjacent to *A*. There are seven color choices for B, but there are only three locations to which B can be rotated to (since there are three edges from A). The remaining six vertices can be colored in any way and their locations are now fixed. Thus the total number of ways is $\frac{8}{8} \times \frac{7}{3} \times 6! = 1680$.

Though the cube may be easier to think about, the octahedron can be directly considered. Since the octahedron is indistinguishable by rotations, without loss of generality fix a face to be red.



There are 7! ways to arrange the remaining seven colors, but there still are three possible rotations about the fixed face, so the answer is $7!\,/\,3 = 1680$.



> *8. For $\{1,2,3,...n\}$ and each of its nonempty subsets a unique alternating sum is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (e.g. the alternating sum for $\{1,2,4,6,9\}$ is $9 - 6 + 4 - 2 - 1 = 6$ and for {5} it is just 5.) Find the sum of all such alternating sums for n = 7.*

Solution 1

Let *S* be a non-empty subset of $\{1, 2, 3, 4, 5, 6\}$.
Then the alternating sum of *S* plus the alternating sum of *S* with 7 included is 7. This is true because when we take an alternating sum, each term of *S* has the opposite sign of each corresponding term of the set *S* with 7 included.
Because there are 63 of these pairs, the sum of all possible subsets of our given set is 63×7.
However, we forgot to include the subset that only contains 7, so our answer is $64 \times 7 = 448$.

Solution 2

Consider a given subset *T* of *S* that contains 7; then there is a subset *T'* which contains all the elements of *T* except for 7, and only those. Since each element of *T'* has one element fewer preceding it than it does in *T*, their signs are opposite; so the sum of the alternating sums of *T* and *T'* is equal to 7. There are 2^6 subsets containing 7, so our answer is $7 \times 2^6 = 448$.