

Here are the comments for Sheet 2. Please encourage mentees, prompt them if they are silent, give them hints as necessary, and be kind and encouraging when you mark and return their work!

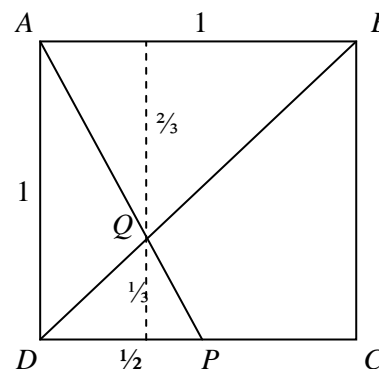
1. Put the digits 1 to 9 (once only each) into the frame so that the product of each three digits is as shown against each row and column.

	6	192	315
20			
126			
144			

One could try an algebraic approach, but it is difficult then to use the fact that the values must be the digits 1 to 9. Instead, probably better to see where the various numbers could go. Probably first consider 5. Only 20 and 315 are multiples of 5, so this must go top right. The only way of writing 20 as a product of three different numbers is to write  $20 = 1 \times 4 \times 5$ , and since 4 does not divide 6, the order must be 1, 4, 5. Now consider the 7. Only 126 and 315 are multiples of 7, so that fixes the 7. Continuing similarly, we find that the rows must be 1 4 5 and 3 6 7 and 2 8 9.

2. ABCD is a square with side length 1. P is the midpoint of CD and Q is the intersection of AP and BD. Find the area of quadrilateral BCPQ.

The key idea here is to recognise that  $\Delta ABQ$  and  $PDQ$  are similar. This of course needs justifying (alternate angles etc). The ratio of lengths is 2 : 1. So the heights  $EQ$  and  $FQ$  as shown are therefore  $\frac{2}{3}$  and  $\frac{1}{3}$ .  $\Delta ABQ$  therefore has area  $\frac{1}{3}$ . Since the area of  $\Delta ADP$  is  $\frac{1}{4}$ , So the area of  $BCPQ$  is  $1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$ .



3. We define the reverse of an integer to be the number obtained by writing its digits in the opposite order. e.g. The reverse of 153 is 351. The two-digit number 72 has the property that multiplying it by  $\frac{3}{8}$  reverses it, since  $\frac{3}{8} \times 72 = 27$ . Find all three-digit numbers and also all four-digit numbers which are reversed by multiplying them by  $\frac{3}{8}$ .

First mentees need to think what the number represented as “abc” actually means, i.e.  $100a + 10b + 10c$ .

We can now formalise that we want a number “abc”, such that:

$$\frac{3}{8} \times (100a + 10b + c) = 100c + 10b + a$$

Multiplying up and simplifying gives:  $292a = 50b + 797c$ .

We see that  $c$  must be even and since  $292 \times 9 = 2628$ ,  $c$  must be less than 4.

Therefore  $c$  can only be 2, and in this case  $292a = 50b + 1594$ .

For  $292a$  to end in 4,  $a$  has to be either 2 (too small) or 7.

If  $a = 7$ , then  $50b = 450$ , so  $b = 9$  will work.

So the only possibility is that “abc” is 792, and we note that  $792 \times \frac{3}{8} = 297$ .

The same principle can be used for a *four*-digit number “ $abcd$ ”.

Putting  $\frac{3}{8} \times (1000a + 100b + 10c + d) = 1000d + 100c + 10b + a$

Giving  $2992a + 220b - 770c = 7997d$

The maximum value of  $2992a + 220b$  is  $2992 \times 9 + 220 \times 8 = 28688$ , so  $d < 4$  and  $d$  is even.

So  $d$  can only be 2, giving  $2992a + 220b - 770c = 15994$ .

As before we can only have  $a = 7$ , giving  $4950 = 770c - 220b$

Which cancels to  $45 = 7c - 2b$

$c$  must be at least 7 and odd. If  $c = 7$ , then  $2b = 4$  so  $b = 2$ .

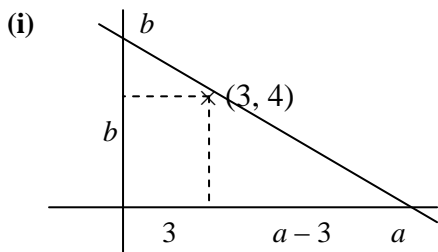
If  $c = 9$ , then  $2b = 18$  so  $b = 9$ .

The possibility  $a = 9$  leads to no further solutions so the only two possible numbers are

7272 and 7992. We note that  $7272 \times \frac{3}{8} = 2727$  and  $7992 \times \frac{3}{8} = 2997$ .

4. A straight line is drawn through the point  $P(3, 4)$ . The line cuts the axes at  $A$  and  $B$  such that the area of triangle  $OAB$  is 24, where  $O$  is the origin. Determine the  $x$ -intercepts of all such lines.

The first thing to note is that there are three possible lines:



In (i),  $\frac{b}{a} = \frac{4}{a-3}$  with  $ab = 48$

So  $ab - 3b = 4a$ .

Putting  $b = 48/a$ ,  $48 - 144/a = 4a$

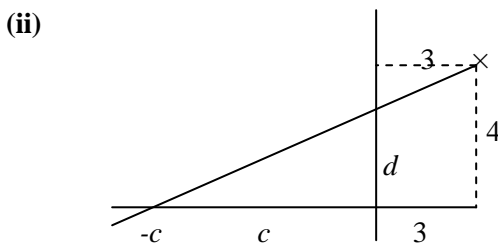
which leads to  $a^2 - 12a + 36 = 0$ ,

So  $(a - 6)^2 = 0$ , and so  $a = 6$ .

So the possible  $x$ -intercepts are

6,  $6(\sqrt{2} + 1)$  and  $6(\sqrt{2} - 1)$ .

This makes me think there was a simpler way, or a way which will do all three cases at once. Offers?



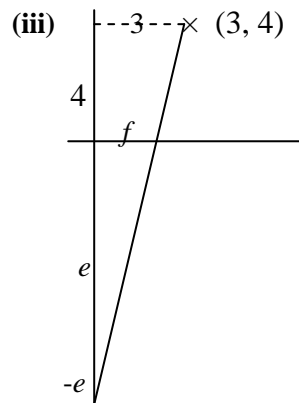
In (ii),  $\frac{d}{c} = \frac{4}{c+3}$  with  $cd = 48$

So  $dc + 3d = 4c$

Putting  $d = 48/c$ ,  $48 + 144/c = 4c$

which leads to  $c^2 - 12c - 36 = 0$

So  $(c - 6)^2 = 72$ , and so  $c = 6\sqrt{2} + 6$ .



In (iii),  $\frac{f}{e} = \frac{3}{e+4}$

giving  $ef + 4f = 3e$ .

Putting  $e = 48/f$ ,

$48 + 4f = 144/f$

leading to  $f^2 + 12f - 36 = 0$

so  $(f + 6)^2 = 72$ .

So  $f = 6\sqrt{2} - 6$ .

5. Find all quadruples of integers  $(a, b, c, d)$  such that  $52^a \times 77^b \times 88^c \times 91^d = 2002$ .

Well, hopefully any mentee who knows anything will realise that you need to change everything into prime factors, and that's about all there is to this question!

You get  $(2^2 \times 13)^a \times (7 \times 11)^b \times (2^3 \times 11)^c \times (7 \times 13)^d = 2 \times 7 \times 11 \times 13$

So we have  $2^{2a+3c} \times 7^{b+d} \times 11^{b+c} \times 13^{a+d} = 2 \times 7 \times 11 \times 13$

So (1)  $2a + 3c = 1$ , (2)  $b + d = 1$ , (3)  $b + c = 1$ , (4)  $a + d = 1$

From (2) and (3),  $c = d$  and from (2) and (4),  $a = b$ , so we have  $2a + 3c = 1$  and  $a + c = 1$ .

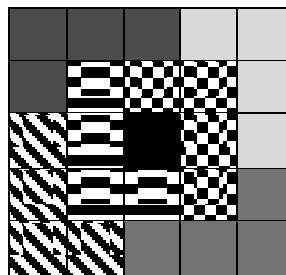
Hence  $c = -1$ ,  $a = 2$ , and  $b = 2$  and  $d = -1$ .

So the only quadruple is  $(2, 2, -1, -1)$  and the result says:  $\frac{52^2 \times 77^2}{88 \times 91} = 2002$ .

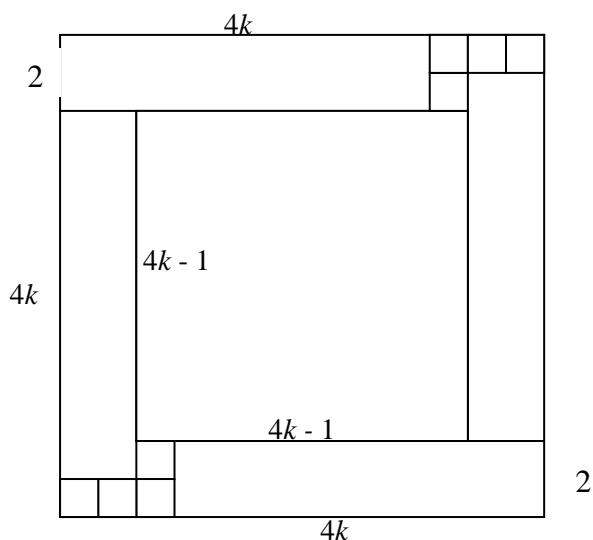
6. Is it possible to cover an  $n \times n$  chessboard with its centre square cut out with tiles shown in the figure opposite if a)  $n = 5$  b)  $n = 2011$  ?  
 [The tiles may be slid into any orientation, but not turned upside down.]

For (a), I think you have just got to try it and hopefully

come up with the idea:



For (b), we need some way of generating a shading given a previous one. If you can do a  $(4k - 1) \times (4k - 1)$  chessboard (without its middle square), you can also do a border of width 2 as follows:



This relies on noting that you can tile a  $2 \times 4k$  rectangle, because you can tile a  $2 \times 4$  rectangle with just two of the L-shaped tiles.

So we can take any  $(4k - 1) \times (4k - 1)$  chessboard and enlarge it to a tiled  $(4k + 3) \times (4k + 3)$  chessboard by adding such a border. Since we can do a  $3 \times 3$  board, we can therefore do a  $7 \times 7$  board, and hence an  $11 \times 11$  board, and so on. By doing the construction 502 times to a  $3 \times 3$  board, we can do a  $2011 \times 2011$  board.

7. Find all possible integer values of  $\frac{m^2 + n^2}{mn}$  where  $m$  and  $n$  are integers.

This is a typical number theory question which needs an approach to do with divisibility and prime factors. This would be the first hint to give to mentees.

Firstly, since  $mn$  divides  $m^2 + n^2$ , we conclude that  $m$  must divide  $n^2$ , and also that  $n$  must divide  $m^2$ .

So  $m$  and  $n$  must have the same prime divisors.

Take a particular prime  $p$  which divides both  $m$  and  $n$ .

Suppose  $m = p^a m'$  and  $n = p^b n'$ , where neither  $m'$  nor  $n'$  is divisible by  $p$ .

Suppose that  $a > b$ .

Then  $mn$  and  $m^2$  are both divisible by  $p^{a+b}$  but  $n^2$  is not.

That is,  $(m^2 + n^2)/(mn)$  is not an integer - a contradiction.

A similar argument follows if  $a < b$ .

So we must have  $a = b$ , i.e. the highest power of  $p$  which divides  $m$  is the same as the highest power of  $p$  which divides  $n$ .

This is true for any prime factor of  $m$  and  $n$ .

Consequently we either have  $m = n$  or  $m = -n$ . So  $\frac{m^2 + n^2}{mn} = \frac{2n^2}{\pm n^2} = \pm 2$ .

8. Find all possible sets of three consecutive positive integers such that each of the three integers is either a prime or a power of a prime.

I suppose mentees should be encouraged to mess around a bit and then consider what primes could be involved. Firstly, since at least one of the three integers must be divisible by 2, and hence it must be a power of 2. If two of them are powers of 2, these must differ by 2, so must be 2 and 4, and the number between is 3, so this is a possible triple.

The only other possibility is that the middle number is a power of 2. If it is 2, we get 1, 2, 3 which is allowable since  $1 = p^0$  for any prime (it may be mentees will miss this – a trivial oversight to my mind). If the middle number is 4, then we have 3, 4, 5 which works. If the middle number is 8, we have 7, 8, 9 which also works.

We now show that there are no other possible solutions. To do this, you really need to consider remainders, so mentees would be well advised to consider modular arithmetic, and they might well be directed to Vicky Neale's excellent introduction to this subject which has a link from the Mentoring website. So they can either go to <http://www.ukmt.org.uk> and go to mentoring and follow the link, or go direct to the article at <http://nrich.maths.org/4350>.

Suppose the middle number is  $2^k$ , where  $k > 3$ , so the numbers are  $2^k - 1$ ,  $2^k$ ,  $2^k + 1$ . As one of the three numbers must be divisible by 3, either  $2^k - 1$  or  $2^k + 1$  would have to be a power of 3. We consider these two cases separately.

It is now helpful to use modular arithmetic (the arithmetic of remainders).

For example considering remainders mod 8, we note that the powers of 3 are all equivalent to either 1 or 3 (mod 8), since  $3^1 = 3$ , and  $3^2 = 9 \equiv 1 \pmod{8}$ , and hence  $3^3 \equiv 3 \pmod{8}$ , etc.

1) If  $2^k - 1 = 3^n$ , but  $2^k - 1 \equiv 7 \pmod{8}$  since  $k > 3$ , and no power of 3 can be  $\equiv 7 \pmod{8}$ .

2) Now suppose that  $2^k + 1 = 3^n$ . Consider the sides of this equation modulo 4.  $2^k + 1 \equiv 1 \pmod{4}$ , and we note that  $3^n \equiv 1 \pmod{4}$  only for even  $n$ , since if  $n$  is odd then  $3^n \equiv 3 \pmod{4}$ . So  $n$  would have to be even.

Now consider the sides of the equation modulo 7.

The even powers of 3 (i.e. the powers of 9!!) go 2, 4, 1, 2, 4, 1

$$\text{i.e. } 3^{6z} \equiv 1, \quad 3^{6z+2} \equiv 2, \quad \text{and} \quad 3^{6z+4} \equiv 4$$

The powers of 2 mod 7 also go: 2, 4, 1, 2, 4, 1 etc, so adding 1 to these we have:

$$2^{3y} + 1 \equiv 2, \quad 2^{3y+1} + 1 \equiv 3, \quad 2^{3y+2} + 1 \equiv 5$$

The only way therefore that a power of 3 can be equal to 1 more than a power of 2 is for them **both** to be equivalent to 2 mod 7.

So since  $2^k + 1 \equiv 2 \pmod{7}$ , the first number of the triple,  $2^k - 1$ , must be a power of 7.

So we must have  $2^k - 1 = 7^m$ .

Now consider the sides of this equation mod 16. The left hand side  $\equiv 15 \pmod{16}$  since  $k \geq 4$ , but the powers of 7 are alternately 1 and 7 (mod 16), so cannot  $\equiv 15$ .

So there are no other possible triples.

The only solutions are (1, 2, 3), (2, 3, 4), (3, 4, 5) and (7, 8, 9).

Of course the difficulty is not in getting these but in proving that there are no others, and mentees will learn quite a bit about modular arithmetic in the process of doing this.

*I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to [mentoring@ukmt.org](mailto:mentoring@ukmt.org) is of course also very welcome.*