

BMOS Mentoring Scheme (Intermediate Level)

Sheet 1 - Solutions and Comments

October 2006

Note that these are only examples of solutions: there are several ways of doing (at least some of) these questions.

1. What is the remainder when 2006^{2006} is divided by 7?

Answer: 2.

We shall use the notation of modular arithmetic. We write $a \equiv b \pmod n$ precisely when n divides $a - b$. That is, a and b leave the same remainder on division by n . If you have not met this idea before, you might like to read a little more about it, for example on the NRICH website (http://www.nrich.maths.org/public/viewer.php?obj_id=4350).

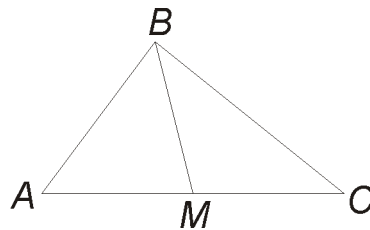
The question is asking us to find $2006^{2006} \pmod 7$.

We can first notice that $2006 = 286 \times 7 + 4$, so $2006 \equiv 4 \pmod 7$. So $2006^{2006} \equiv 4^{2006} \pmod 7$.

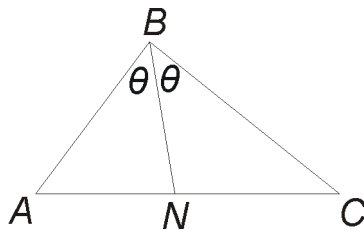
Now let us think about powers of 4 mod 7. We have $4^1 \equiv 4 \pmod 7$, $4^2 \equiv 2 \pmod 7$, and $4^3 \equiv 1 \pmod 7$. Now this cycle just keeps repeating. So the value of $4^n \pmod 7$ depends on the value of $n \pmod 3$. We have $2006 = 668 \times 3 + 2$, so $2006 \equiv 2 \pmod 3$, so $4^{2006} \equiv 4^2 \equiv 2 \pmod 7$.

A useful and relevant theorem here is *Fermat's Little Theorem* (Little rather than Last!). It says that if p is a prime and a is not divisible by p then $a^{p-1} \equiv 1 \pmod p$. You might like to look up a proof.

2. (i) Let ABC be a triangle. Let M be the midpoint of the side AC . Show that the areas of triangles ABM and MBC are equal.
- (ii) Let ABC be a triangle. Let N be the point of intersection of the side AC with the bisector of angle B . Show that $AB : BC = AN : NC$.
- [This is known as the *angle bisector theorem*.]



- (i) Let h denote the perpendicular distance from B to the line AC . Then triangle ABM has area $\frac{1}{2}h \times AM$, and triangle MBC has area $\frac{1}{2}h \times MC$. But M is the midpoint of the sides AC , so $AM = MC$, so triangles ABM and MBC have the same area.



(ii) Let θ denote $\angle ABN$ (this notation means angle ABN), so also $\theta = \angle NBC$.

Then the area of triangle ABN is $\frac{1}{2} \times AB \times BN \times \sin \theta$. (This is a standard formula, which it is useful to remember. If you have not met it before, you might like to try to prove it, using the ' $\frac{1}{2} \times \text{base} \times \text{height}$ ' formula.)

Similarly, the area of triangle NBC is $\frac{1}{2} \times BN \times BC \times \sin \theta$.

Thus the ratio of the area of triangle ABN to the area of triangle NBC is $\frac{1}{2} \times AB \times BN \times \sin \theta : \frac{1}{2} \times BN \times BC \times \sin \theta$, that is, $AB : BC$.

Using a very similar argument to that used in part (i), we see that the ratio of the area of triangle ABN to the area of triangle NBC is $AN : NC$ (the triangles have the same perpendicular height).

So $AB : BC = AN : NC$.

3. Find all real numbers x such that

$$\sqrt{x-10} + \sqrt{x-20} = \sqrt{x+22}.$$

Answer: The only solution is $x = 28$.

Note that by convention \sqrt{x} means the non-negative square root of x .

Suppose that x is a real number such that $\sqrt{x-10} + \sqrt{x-20} = \sqrt{x+22}$.

Then, squaring, we have $x-10 + 2\sqrt{(x-10)(x-20)} + x-20 = x+22$. At this stage, we may have introduced some extra solutions, by squaring. We shall check this at the end.

Simplifying, we get $2\sqrt{x^2 - 30x + 200} = 52 - x$, and squaring again we get $4x^2 - 120x + 800 = 2704 - 104x + x^2$, so $3x^2 - 16x - 1904 = 0$.

We may factorise this as $(3x+68)(x-28) = 0$, so either $x = -\frac{68}{3}$, or $x = 28$.

It is important to be clear what we have done. We have shown that *if* x satisfies $\sqrt{x-10} + \sqrt{x-20} = \sqrt{x+22}$, *then* $x = -\frac{68}{3}$ or $x = 28$. We have *not* shown that these values of x satisfy the equation, because squaring is not reversible (there are two square roots). This is the same as the fact that *if* $x = 1$ *then* $x^2 = 1$, but we cannot conclude from $x^2 = 1$ that $x = 1$ (we could have $x = -1$).

However, that does not mean that we have wasted our time. We know that if x satisfies the equation then either $x = -\frac{68}{3}$ or $x = 28$, and it is easy to check whether or not these satisfy the equation. We find that $x = 28$ does; $x = -\frac{68}{3}$ does not.

4. A postman has seven letters to deliver, one to each of the seven dwarves. Unfortunately, he is rather inaccurate in making his deliveries.

- (i) In how many ways could he deliver the letters so that exactly two of the dwarves get the wrong letters?
- (ii) In how many ways could he deliver the letters so that exactly three of the dwarves get the wrong letters?

Answer: (i) 21 ways; (ii) 70 ways.

- (i) If exactly two dwarves get the wrong letters, then they have each got the other's letter. So we must count the number of possible pairs of unlucky dwarves, and this will give the answer.

There are 7 ways to choose the first unlucky dwarf, and once he has been selected there are 6 possibilities for the second dwarf. This gives $7 \times 6 = 42$ pairs. However, we have counted each pair precisely twice (Grumpy then Sneezzy and Sneezzy then Grumpy, for example). So we must divide by 2, to give $\frac{7 \times 6}{2} = 21$.

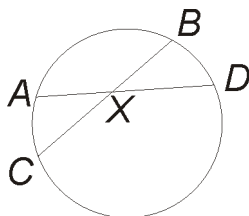
- (ii) Let us first count the number of possible trios of unlucky dwarves.

There are 7 ways to choose the first dwarf, 6 ways to choose the second, and 5 ways to choose the third, giving $7 \times 6 \times 5 = 210$. Again, however, we have counted each trio several times; this time we have counted them each $3 \times 2 = 6$ times. So there are $\frac{7 \times 6 \times 5}{3 \times 2} = 35$ possible trios.

Now, in how many ways can a particular trio (say dwarves A , B and C) get the wrong letters? Dwarf A must get the letter for B or C — two possibilities. If he gets B 's letter, then C must get A 's letter and B gets C 's. Similarly, if A gets C 's letter, then B gets A 's and C gets B 's. So there are two possible ways.

Thus the total number of ways in which the postman could deliver the letter so that exactly three of the dwarves get the wrong letter is $35 \times 2 = 70$.

5. Let AD and BC be two chords of a circle that intersect inside the circle at the point X . Show that $(AX)(DX) = (BX)(CX)$.



Remember that $\angle ABC$ means angle ABC (that is, the angle between AB and BC).

We shall use the “angles in the same segment” theorem. If you have not come across it, you should look it up — it is one of the standard circle theorems and is very useful.

For example, here we have $\angle CAX = \angle XBD$ — they are on the same side of the chord CD and are both subtended by CD .

Similarly, we have $\angle ACX = \angle XDB$.

Also, we have $\angle AXC = \angle BXD$ (angles between intersecting lines).

Thus we see that triangles ACX and BDX are similar, as they have the same angles.

From this, we deduce that $\frac{AX}{CX} = \frac{BX}{DX}$ (ratios of corresponding sides), so $(AX)(DX) = (BX)(CX)$.

6. Find all positive integers x that leave a remainder of 1 when divided by 2, 3 and 5.

Answer: All numbers of the form $30k + 1$ where k is an integer.

If x leaves a remainder of 1 when divided by 2, 3 and 5, then $x - 1$ is divisible by 2, 3 and 5.

Since 2, 3 and 5 have no common factors, $x - 1$ must be divisible by $2 \times 3 \times 5 = 30$. So x is 1 more than a multiple of 30, i.e., $x = 30k + 1$ for some integer k .

If $x = 30k + 1$, then $x = (2 \times 15k) + 1 = (3 \times 10k) + 1 = (5 \times 6k) + 1$, so x leaves a remainder of 1 when divided by 2, 3 and 5.

Thus the solutions are all numbers of the form $30k + 1$ where k is an integer.

7. A *tetromino* is a shape made up of four unit squares joined along their edges. We count two tetrominoes as the same if one is a rotation of the other.
- (i) Show that there are exactly seven different tetrominoes.
- (ii) Is it possible to cover a 4×7 rectangle using each tetromino precisely once (without overlapping)?

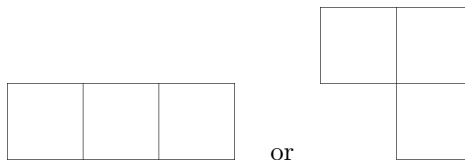
Answer: (ii) No.

- (i) There is only one shape made up of two squares:

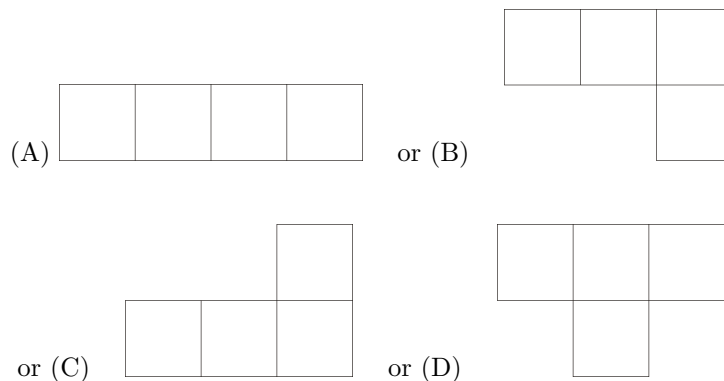


(the others are all rotations of this).

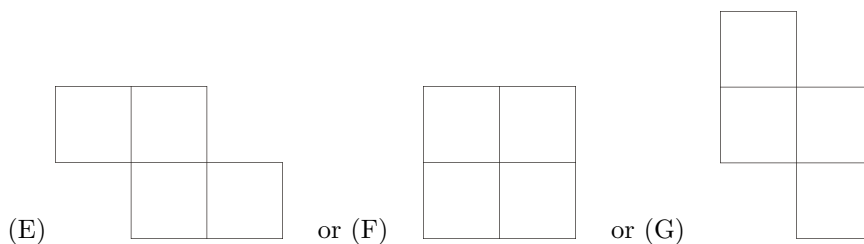
We may add a square on the end or on the side:



We can add a square to the first of those in four ways:



We can add a square to the second of the shapes with three squares in three ways:



These are therefore the only tetrominoes, and there are seven of them.

- (ii) Colour the squares of the rectangle black and white, like on a chessboard. Notice that there are equal numbers of black and white squares.
- Each of shapes A, B, C, E, F and G covers two squares of each colour (regardless of where it is positioned on the board). However, shape D covers three squares of one colour and one of

the other. So using all the tetrominoes precisely once would cover more squares of one colour than the other, so they cannot be used to cover the 4×7 board.

8. Find all positive integers a and b such that

$$\frac{4}{ab} + \frac{2}{a} + \frac{1}{b} = 1.$$

Answer: $a = 8, b = 2$, or $a = 5, b = 3$, or $a = 4, b = 4$, or $a = 3, b = 7$.

Suppose that a and b are positive integers satisfying $\frac{4}{ab} + \frac{2}{a} + \frac{1}{b} = 1$.

Then, multiplying by ab , we have $4 + a + 2b = ab$, so $ab - a - 2b = 4$. We may write this as $(a - 2)(b - 1) = 6$.

Since a and b are integers, we know that $a - 2$ and $b - 1$ must be factors of 6.

Also, we know that a and b are positive, so $a - 2 \geq -2$ and $b - 1 \geq -1$, so we only need to look at positive factors of 6.

Case 1: $a - 2 = 6, b - 1 = 1$. Then $a = 8, b = 2$.

Case 2: $a - 2 = 3, b - 1 = 2$. Then $a = 5, b = 3$.

Case 3: $a - 2 = 2, b - 1 = 3$. Then $a = 4, b = 4$.

Case 4: $a - 2 = 1, b - 1 = 6$. Then $a = 3, b = 7$.

It is easy to check that all of these are indeed solutions.