BMOS Mentoring Scheme (Intermediate Level)

Sheet 1 - October 2006

These questions are not necessarily in order of difficulty, and you do not have to attempt them in order.

1. What is the remainder when 2006^{2006} is divided by 7?

2. (i) Let ABC be a triangle. Let M be the midpoint of the side AC. Show that the areas of triangles ABM and MBC are equal.

   (ii) Let ABC be a triangle. Let N be the point of intersection of the side AC with the bisector of angle B. Show that $AB : BC = AN : NC$.
   [This is known as the *angle bisector theorem*.]

3. Find all real numbers x such that
$$\sqrt{x-10} + \sqrt{x-20} = \sqrt{x+22}.$$

4. A postman has seven letters to deliver, one to each of the seven dwarves. Unfortunately, he is rather inaccurate in making his deliveries.

   (i) In how many ways could he deliver the letters so that exactly two of the dwarves get the wrong letters?

   (ii) In how many ways could he deliver the letters so that exactly three of the dwarves get the wrong letters?

5. Let AD and BC be two chords of a circle that intersect inside the circle at the point X. Show that $(AX)(DX) = (BX)(CX)$.

6. Find all positive integers x that leave a remainder of 1 when divided by 2, 3 and 5.

7. A *tetromino* is a shape made up of four unit squares joined along their edges. We count two tetrominoes as the same if one is a rotation of the other.

   (i) Show that there are exactly seven different tetrominoes.

   (ii) Is is possible to cover a 4×7 rectangle using each tetromino precisely once (without overlapping)?

8. Find all integers a and b such that
$$\frac{4}{ab} + \frac{2}{a} + \frac{1}{b} = 1.$$

Deadline: 27th October 2006