BMOS MENTORING SCHEME (Intermediate Level)
April 2012 (Sheet 7)
Comments for Mentors and Teachers

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1. An engineer can finish a highway in three days with his present supply of machines. With three more machines the job can be done in two days. If the machines all work at the same rate, how many days would it take to do the job with one machine?

Suppose a single machine would take d days and that the engineer initially has m machines.
Then we know that $d/m = 3$, and also that $d/(m + 3) = 2$. So $d = 3m = 2m + 6$. So $m = 6$, and $d = 18$.
A single machine would take 18 days.

2. Find all real values of x and y for which $x + y + xy = –5$ and $x^2 + y^2 + x^2y^2 = 49$.

I don't know how many different ways there are of doing this but the thing that occurs to me is that the terms in the second equation are the squares of the terms in the first. So I will try squaring the first and then substituting:

$$(x + y + xy)^2 = x^2 + y^2 + x^2y^2 + 2(xy + x^2y + xy^2),$$

so we have $25 = 49 + 2xy(1 + x + y)$,
and hence $xy(1 + x + y) = –12$.
Now if we substitute for $x + y$ from the first equation, we will get an equation in xy:

$$xy(1 – 5 – xy) = –12,$$

so $(xy)^2 + 4xy – 12 = 0$, which gives $(xy + 6)(xy – 2) = 0$, so $xy = 2$ or $–6$.
Now we still have a bit of work to do.

If $xy = –6$, we have $x + y = 1$ and $x^2 + y^2 = 13$, so $x^2 –2xy + y^2 = 25$, so $(x – y)^2 = 25$ and hence $x – y = \pm 5$.
Solving with $x + y = 1$ gives $(x, y) = (3, –2)$ or $(–2, 3)$

If $xy = 2$, we have $x + y = –7$ and $x^2 + y^2 = 45$, so $x^2 –2xy + y^2 = 41$, so $(x – y)^2 = 41$ and hence $x – y = \pm\sqrt{41}$.

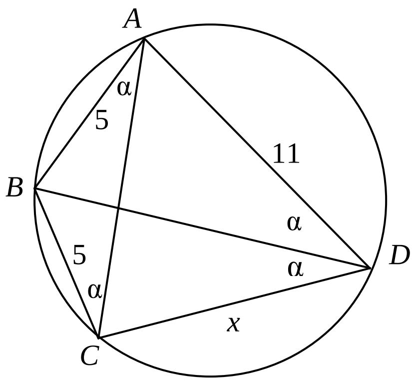
Solving with $x + y = –7$, we get $(x, y) = \left(\frac{-7+\sqrt{41}}{2}, \frac{-7-\sqrt{41}}{2}\right)$ or $\left(\frac{-7-\sqrt{41}}{2}, \frac{-7+\sqrt{41}}{2}\right)$

So there are four points of intersection of the two curves, which can be shown using Autograph or Geogebra.

3. Lines L_1, L_2, ... L_{100} are all distinct. All lines L_{4n}, where n is a positive integer, are parallel to each other. All lines $L_{4k – 3}$, where k is a positive integer, pass through a given point A. Determine the maximum number of points of intersection of pairs of lines from the complete set $\{L_1, L_2, ... L_{100}\}$.

There are probably two ways to approach this question. Either count up the number of ways each type of line meets all the other types of lines, or to count the total number of possible intersections and subtract the ones that can't happen. I prefer the latter approach, and will do this now. In the full solutions, I will try to remember to include the other approach.
If the 100 lines were all totally random (but no two parallel), then each pair of lines would have an intersection, so the number of intersections would be ${}^{100}C_2 = 100 \times 99 / 2 = 4950$.
But the 25 "L_{4n}" lines are all parallel, so we lose ${}^{25}C_2 = 25 \times 24 / 2 = 300$ intersections for these lines.
And the 25 "$L_{4n–3}$" lines all meet at one point, so we lose ${}^{25}C_2$ intersections, but gain 1.
Thus the total number is ${}^{100}C_2 – {}^{25}C_2 – {}^{25}C_2 + 1 = 4950 – 300 – 300 + 1 = 4351$.

4.	*The circumcircle of cyclic quadrilateral ABCD has diameter 5√5, with AB = 5, BC = 5 and CD = 11. How long is AD?*



It is difficult to know where to start with this question, but maybe after several false starts, you think that the two 5's must be of some significance. One is tempted to draw in a diameter or some radii, but I think this is probably unhelpful. One thought is to use the fact that the circle is the circumcircle to several Δs, so the (full) Sine Rule could help.
If we let *BAC* = *α*, we note that *BCA* = *α* (isos Δ), and also *BDA* and *CDB* are also both α (angles in same segment).
Since in Δ*ABC*, by the Sine Rule, 5/sin *α* = 2*R* = 5√5, sin *α* = 1/√5, so α is the angle in a 1, 2, √5 Δ, and cos α = 2/√5.
Now we can get two expressions for side *AC*.
From isos Δ*ABC*, we see that *AC* = 2 × 5cos α = 4√5.
And using the Cosine Rule in Δ*ADC*, $AC^2 = x^2 + 11^2 - 22x\cos(2\alpha)$. (*)
Those who know that $\cos(2\alpha) = 1 - 2\sin^2\alpha$ are at an advantage here, but alternatively we can get a value for cos (2α) by reflecting the 1, 2, √5 Δ on the 2 side, and using the Cosine Rule to get
$\cos(2\alpha) = (5 + 5 - 2^2) / (2 \times 5) = 3/5$, so substituting in (*) we get $(4\sqrt{5})^2 = x^2 + 11^2 - 22x \times 3/5$,
leading to $5x^2 - 66x + 205 = 0$. Hence $(5x - 41)(x - 5) = 0$, so $x = 5$ or 8.2.
So there are two possible values for *CD*, namely 5 and 8.2, according to which side *AD* the centre lies.
[If we drew a circle centre *A*, radius 11, it would meet the circle *ABC* in two places.]

5.	*Show that all the primes except 2 and 3 occur as terms of the sequence defined by* $a_n = \sqrt{24n+1}$ *for n = 1, 2, 3, ...*

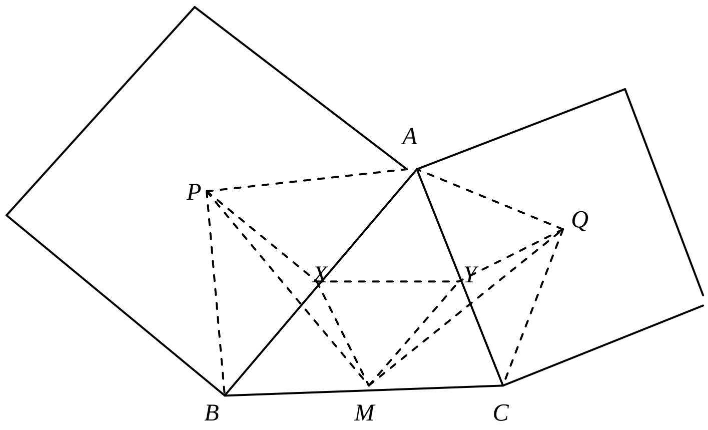
I suppose as usual, the thing to do is to try some values of *n*. If we work through from *n* = 1, we get:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	**5**	**7**	√73	√97	**11**	√145	**13**	√193	√217	√241	√265	**17**	√313	√337	**19**

So we see that we do indeed get the first few primes after 3. What is going on here?
There is a useful fact, which is actually suggested by these results above, and that is that the primes are either side of a multiple of 6.
After a few moments' thought, this is fairly obvious since primes, being odd (apart from 2), must either be 6*n* + 1, 6*n* + 3, or 6*n* + 5. But 6*n* + 3 is divisible by 3, so this is the only prime of this form.
(And of course numbers 5 more than a multiple of 6 are the same as numbers 1 less than a multiple of 6.) So useful fact: All primes apart form 2 and 3 are of the form 6*n* ± 1.
So if we can show that all such numbers are produced by a_n for some number *n*, then we are done.

If $6k+1=\sqrt{24n+1}$, then $(6k + 1)^2 = 24n + 1$, so $36k^2 + 12k = 24n$, and $3k^2 + k = 2n$,
so $n = \frac{1}{2} k (3k + 1)$, and since either *k* or 3*k* + 1 is even, *n* will always be an integer, so for every *k* we can find an integer *n* such that $6k+1=\sqrt{24n+1}$.
Similarly if $6k-1=\sqrt{24n+1}$, we find that $n = \frac{1}{2} k (3k - 1)$, and by similar reasoning, for every *k* we can find an integer *n* such that $6k-1=\sqrt{24n+1}$.
Thus we can find integers *n* to give **<u>all</u>** numbers of the form 6*k* ± 1, and this includes all the primes except for 2 and 3.

6. *P and Q are the centre of two squares drawn on the sides AB and AC of ΔABC. If M is the midpoint of BC, prove that ΔMPQ is isosceles and right-angled.*



We are asked to prove that $PM = MQ$ and the $PMQ = 90^{o}$.
This looks simple, but it is not! It is difficult to know what to draw or what strategy to use. Congruent triangles are hard to find, and yet you feel this is what is needed.
If we mark the midpoints of *AB* and *AC*, and call them *X* and *Y*, it is well known that *XY* is parallel to *BC* etc, and joining the midpoints of the sides splits Δ*ABC* into four congruent triangles.
So $YM = XB = XP$ and $XM = YC = YQ$.
Also $PXM = 90 + A = QYM$.
So Δ*PXM* is congruent to Δ*MYQ* and hence $PM = MQ$.

If we let $XPM = \alpha$, then $YMQ = \alpha$,
and $PMB = 180 - (45 - \alpha) - 45 - B = 90 + \alpha - B$.
Whereas $CMQ = B - \alpha$.
So $PMB + CMQ = 90$, and hence $PMQ = 90^{o}$ as required.

7. *Let a, b, c be positive integers such that* $\dfrac{a\sqrt{2}+b}{b\sqrt{2}+c}$ *is a rational number. Prove that* $a + b + c$ *is a divisor of* $a^2 + b^2 + c^2$.

The first thing is to see what we can deduce from the fact that $\dfrac{a\sqrt{2}+b}{b\sqrt{2}+c}$ is rational. Rationalising the denominator by multiplying by $b\sqrt{2} - c$ gives $\dfrac{(a\sqrt{2}+b)(b\sqrt{2}-c)}{(b\sqrt{2}+c)(b\sqrt{2}-c)} = \dfrac{2ab-bc+\sqrt{2}(b^2-ac)}{2b^2-c^2}$,
so since this is rational, we know that $b^2 - ac = 0$, so $b^2 = ac$.

Now if we square $a + b + c$, we get $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$,
so $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = (a + b + c)^2 - 2(ab + bc + b^2)$ since $b^2 = ac$.
So $a^2 + b^2 + c^2 = (a + b + c)^2 - 2b(a + c + b) = (a + b + c)(a - b + c)$,
so $a + b + c$ is a divisor of $a^2 + b^2 + c^2$, as required.

8.	*The number 12 may be factored into three numbers in eighteen ways. These factorisations include $1 \times 3 \times 4$, $2 \times 2 \times 3$, $2 \times 3 \times 2$ and fifteen other examples. Let N be the number of seconds in a week. In how many ways can N be factored into three positive integers?*

Firstly it is very helpful to see if we agree with the first statement that 12 can be factorised if 18 ways.
$12 = 2^2 \times 3$. So (since we see 1's are allowed) it can be factorised as follows:

with two 1's	$1 \times 1 \times 12$	in 3 orders
with one 1	$1 \times 2 \times 6$	in 6 orders
	$1 \times 4 \times 3$	in 6 orders
with no 1's	$2 \times 2 \times 3$	in 3 orders, making 18 in total.

Now the number of seconds in a week is $7 \times 24 \times 60 \times 60$, i.e. $N = 2^7 \times 3^3 \times 5^2 \times 7$.
This is rather more unwieldy!

With two 1's	$1 \times 1 \times N$	in 3 orders
With one 1	starting to panic...! though not too bad!	
	the number of factors of N is $8 \times 4 \times 3 \times 2 = 192$ and these split into 96 pairs	
so	$1 \times a \times b$	96 pairs in 6 orders each
With no 1	Ugh help! $a \times b \times c$ but there are lots of combinations.	

Maybe we should go back and see if we can find a simpler way of looking at the factorisations of 12.
$12 = 2^2 \times 3$. Effectively we want to split up the two 2's in an ordered triple and also the single 3.
If you do this you see that there are ${}^4C_2$ ways of splitting the 2's, and ${}^3C_1$ ways of allocating the 3.
So the number of factorisations is ${}^4C_2 \times {}^3C_1 = 6 \times 3 = 18$.

This is perhaps easier to explain using the larger number $N = 2^7 \times 3^3 \times 5^2 \times 7$.
We want to split into 3 (ordered) factors.
Note that the prime factors can be dealt with independently.
So the seven 2's could be split in loads of ways, e.g. 1, 2, 4 or 0, 2, 5 or 7, 0, 0.
The simplest way to see how to partition 7 in to an ordered triple is to visualise it as seven dots with two separators, so for example: ● ● | ● ● ● ● | ● represents 2, 4, 1
and ● ● ● ● ● ● ● | | represents 7, 0, 0.
So the number of ways of partitioning 7 into 3 numbers is the number of ways of arranging the 9 symbols consisting of 7 dots and 2 separators, which is ${}^9C_2$.
Similarly the three 3's can be allocated in ${}^5C_2$ ways (independently of the 2's), e.g. ● ● | | ●
So the number of ordered triples of factors is ${}^9C_2 \times {}^5C_2 \times {}^4C_2 \times {}^3C_2 = 36 \times 10 \times 6 \times 3 = 6480$.

As an illustration to try to make it as clear as I can, I will just do one allocation as an example:

If we allocate the 2's:	2	4	1
and the 3's:	1	2	0
and the 5's:	2	0	0
and the 7:	0	0	1
the three factors are:	$2^2 \times 3 \times 5^2$	$2^4 \times 3^2$	2×7

I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to mentoring@ukmt.org *is of course also very welcome.*

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