UKMT MENTORING SCHEME (Intermediate Level)
April 2012 (Sheet 7)
Questions

1. An engineer can finish a highway in three days with his present supply of machines. With three more machines the job can be done in two days. If the machines all work at the same rate, how many days would it take to do the job with one machine?

2. Find all real values of x and y for which
$$x + y + xy = -5$$
$$x^2 + y^2 + x^2y^2 = 49$$

3. Lines L_1, L_2, ... L_{100} are all distinct. All lines L_{4n}, where n is a positive integer, are parallel to each other. All lines L_{4k-3}, where k is a positive integer, pass through a given point A. Determine the maximum number of points of intersection of pairs of lines from the complete set $\{L_1, L_2, ... L_{100}\}$.

4. The circumcircle of cyclic quadrilateral $ABCD$ has diameter $5\sqrt{5}$, with $AB = 5$, $BC = 5$ and $CD = 11$. How long is AD?

5. Show that all the primes except 2 and 3 occur as terms of the sequence defined by
$$a_n = \sqrt{24n+1} \quad \text{for } n = 1, 2, 3, ...$$

6. P and Q are the centre of two squares drawn on the sides AB and AC of ΔABC. If M is the midpoint of BC, prove that ΔMPQ is isosceles and right-angled.

7. Let a, b, c be positive integers such that $\dfrac{a\sqrt{2}+b}{b\sqrt{2}+c}$ is a rational number. Prove that $a + b + c$ is a divisor of $a^2 + b^2 + c^2$.

   [*Note: A rational number is a number expressible as p/q, where p and q are integers.*]

8. The number 12 may be factored into three numbers in eighteen ways. These factorisations include $1 \times 3 \times 4$, $2 \times 2 \times 3$, $2 \times 3 \times 2$ and fifteen other examples. Let N be the number of seconds in a week. In how many ways can N be factored into three positive integers?

Deadline for receipt of solutions: 30th April 2012

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