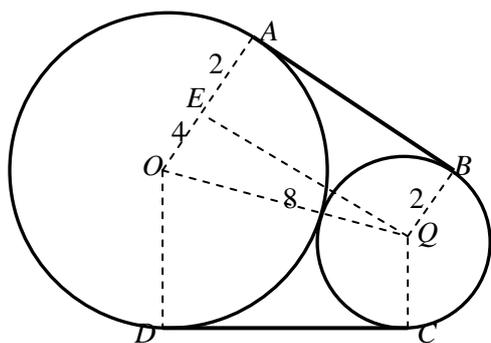


1. The number m is the first of five consecutive integers that are respectively multiples of 3, 4, 5, 6 and 7. If $m > 7$, find the smallest possible value of m .

Clearly a possible set of numbers with this property is 3, 4, 5, 6, 7! However, this is excluded by the condition. So the question is: how far do we have to go until we get the next set? The difference has got to be a multiple of 3, so that we start with a multiple of 3. It must also be a multiple of 4, so that the second number is a multiple of 4. Etc. In other words, the "gap" must be a multiple of all of the numbers, i.e. their Lowest Common Multiple, which is $2^2 \times 3 \times 5 \times 7$, i.e. 420. So the first number m is 423.

2. Two cylinders have diameters of 12 cm and 4 cm respectively. Find the length of the shortest band which can be fixed round the cylinders to hold them together.



In the diagram, we have radii $BQ = 2$ and $AO = 6$, so $OQ = 8$, and if we draw QE parallel to BA , then $OE = 4$ and since angle $OAB = 90^\circ$, $OEQ = 90^\circ$ also, so $\triangle OEQ$ is right-angled, and $EQ = 4\sqrt{3}$, with angle $EOQ = 60^\circ$.
 So the straight sections of the band each have length $4\sqrt{3}$.
 And angle $AOD = 120^\circ$, so the band goes round $\frac{2}{3}$ of the large circle and $\frac{1}{3}$ of the small circle.
 So the length of the band is $8\sqrt{3} + \frac{2}{3} \times 2\pi \times 6 + \frac{1}{3} \times 2\pi \times 2$ which simplifies to $8\sqrt{3} + 28\pi / 3$ cm.

3. The sequence 9, 17, 24, 36 is one in which each term is the sum of the corresponding terms of two other sequences. The first of these is formed after the first two terms by each term being the sum of the previous two terms. The second is an arithmetic sequence (whose consecutive terms have a constant difference). Find the 7th term of the resulting sequence.

The key here is to use algebra. From the information we are given, we could write the sequences as:

$$\begin{array}{cccc} a, & b, & a + b, & a + 2b \\ c, & c + d, & c + 2d, & c + 3d \end{array}$$

Adding these gives the equations:

$$\begin{aligned} a + c &= 9 & (1) \\ b + c + d &= 17 & (2) \\ a + b + c + 2d &= 24 & (3) \\ a + 2b + c + 3d &= 36 & (4) \end{aligned}$$

(4) - (3) gives $b + d = 12$

and combining with (2) gives $c = 5$. So from (1), $a = 4$.

Substituting in (3) gives $b + 2d = 15$, and since $b + d = 12$, we have that $d = 3$ and hence $b = 9$.

So the sequences are:

4	9	13	22	35	57	92	...	
and	5	8	11	14	17	20	23	...

and the final sequence is

9	17	24	36	52	77	115	...
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so the 7th term as required is 115.

4. Find the smallest integer which has precisely 2012 positive integer factors.

For those who have not seen this before, you will need to start by seeing how to find the number of factors of a given integer, so start with a few small numbers which do have several factors, e.g. 72. To do this you should consider the prime factorisation of the number. So $72 = 2^3 \times 3^2$.

Possible factors are given by $2^0 \times 3^0, 2^1 \times 3^0, 2^2 \times 3^0, 2^3 \times 3^0$, i.e. the four choices of powers of 2 with 3^0 .

Then you could have the four choices of powers of 2 with 3^1 and similarly again four with 3^2 .

So we see there are 4 choices for the power of 2 and 3 choices for the power of 3, making 4×3 factors in all.

And for example the number $36000 = 2^5 \times 3^2 \times 5^3$ would have $6 \times 3 \times 4 = 72$ factors.

Armed with this knowledge, we can now address the question.

Possible factorisations of 2012 are $2 \times 2 \times 503, 4 \times 503, 2 \times 1006$, and of course just 2012.

So we need to adjust the powers of (small) primes appropriately.

e.g. 2^{2011} has 2012 factors; so does $2^1 \times 3^{1005}$ but of course $2^{1005} \times 3$ is better, and the best so far.

We could have $2^{502} \times 3^3$, but in fact this is going to be beaten by $2^{502} \times 3^1 \times 5^1$ which will be the winner.

This is way too big for any calculator! And since $2^{10} = 1024 > 10^3$, the winning number is at least 10^{150} !!

5. Find the set of 100 consecutive odd positive integers whose sum is 100^{100} .

Your approach to this will depend on what you know. You are encouraged to investigate the sums of odd numbers, starting small, e.g. $1 + 3 + 5 = 9$, and $1 + 3 + 5 + 7 = 16$, and to realise that the first n odd integers add to n^2 . I distinctly remember this being illustrated to me when I was at school, and the teacher showed L-shapes being added to a diagram effectively giving a nice proof of this result as shown below.

Armed with this knowledge:

Suppose the required 100 numbers are:

$$(2n + 1) + (2n + 3) + \dots + (2n + 199) = 100^{100}.$$

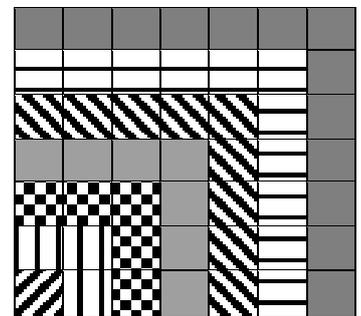
$$\text{So } 200n + 1 + 3 + \dots + 199 = 100^{100}.$$

$$\text{So from the above } 200n + 100^2 = 100^{100}.$$

$$\text{Therefore } 200n = 100^{100} - 100^2$$

$$\text{and hence } 2n = 100^{99} - 100.$$

So the numbers are $100^{99} - 99, 100^{99} - 97, \dots, 100^{99} + 99$.



6. Show that given any 16 composite integers less than 2500, at least two of them will have a prime factor in common.

Well, the first 16 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53 and any composite number less than 2500 must have one factor less than or equal to its square root, so less than 50. It therefore must have a prime factor less than 50. Since there are only 15 primes less than 50, as soon as we bring in a 16th number, it must share a prime factor with one of the other 15 numbers. This is by the pigeon-hole principle, which states that if there are only n pigeon-holes, then if you have $n + 1$ pigeons, at least one pigeon-hole must be occupied by at least two pigeons. Also called Dirichlet's Box Principle.

7. In the right-angled triangle ABC (as shown), CF is the median drawn to the hypotenuse AB , CE is the bisector of the angle ACB , and CD is the altitude to AB (i.e. CD is perpendicular to AB .)
 Prove that $\angle DCE = \angle ECF$.

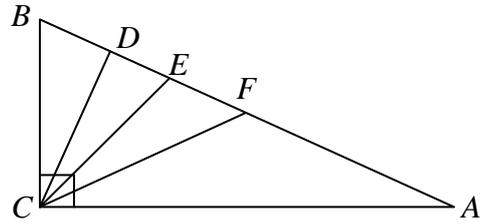
This, I suddenly realise, is not a difficult question!
 We know that $BCE = ACE = 45^\circ$, and we are trying to prove that $DCE = ECF$.

So it is sufficient to prove that $BCD = ACF$, since $DCE = 45^\circ - BCD$ and $ECF = 45^\circ - ACF$.

If $BCD = \alpha$, then $DBC = 90^\circ - \alpha$, and so $BAC = \alpha$.

But since F is the midpoint of AB , it is the centre of circle ABC , so $FC = FA$ as they are both radii, and hence $FCA = \alpha$ (isosceles triangle).

Hence $BCD = ACF$, and therefore $DCE = ECF$ as required.



8. Let x, y be positive real numbers such that $10x + 3y = 35$.
 Find the maximum possible value of $x^4 y^3$.
 [Hint: recall that on last month's sheet we proved (in outline) the "Arithmetic Mean - Geometric Mean" Inequality, which says that for any set of positive numbers a_1, a_2, \dots, a_n , it is true that $\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$, with equality when $a_1 = a_2 = \dots = a_n$.]

This is a fun question, I think, and I hope that the hint to use AM-GM will be enough. You need to think how you might use this to help with what is given.

We are trying to maximise $x^4 y^3$. So it would be good to make that the Right Hand Side of the AM-GM inequality. So for example, we could say that if we take the quantities x, x, x, x, y, y, y , then by the AM-GM inequality, we have $\frac{x + x + x + x + y + y + y}{7} \geq \sqrt[7]{x^4 y^3}$.

But unfortunately we don't know anything about the left side; the quantity $4x + 3y$ is presumably variable. Somehow we want to use the fact that $10x + 3y = 35$, so maybe we could use the quantities $x, 2x, 3x, 4x, y, y, y$, since they add to $10x + 3y$. But the problem is that equality is only attained if all seven of these are the same and this won't be true.

So now you might think of splitting the $10x$ up evenly and considering $5x/2, 5x/2, 5x/2, 5x/2, y, y, y$.

If you do this you get $\frac{\frac{5}{2}x + \frac{5}{2}x + \frac{5}{2}x + \frac{5}{2}x + y + y + y}{7} \geq \sqrt[7]{x^4 y^3 \times (\frac{5}{2})^4}$.

The LHS is just 5, so we get $5^7 \geq \frac{x^4 y^3 \times 5^4}{2^4}$ and therefore $x^4 y^3 \leq 5^3 \times 2^4 = 2000$.

Hence the maximum value attainable by $x^4 y^3$ is 2000 and this occurs when all the seven quantities are equal, i.e. when $y = 5x/2$, so $2y = 5x$, and hence $7y = 35$, so $y = 5$ and $x = 2$, as can be seen above.

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