BMOS MENTORING SCHEME (Intermediate Level)
February 2012 (Sheet 5)
Solutions & Explanatory Notes

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1. Horses X, Y, Z are to run a 3-horse race (in which there are no dead heats). The odds against X winning are 3 - 1, and the odds against Y winning are 2 - 3. What are the odds against Z winning?

Odds against X are 3 - 1, so $prob(X\ wins) = {}^{1}/_{4}$. Odds against Y are 2 - 3, so $prob(Y\ wins) = {}^{3}/_{5}$.
So $prob(X\ or\ Y\ wins) = {}^{1}/_{4} + {}^{3}/_{5} = {}^{17}/_{20}$. Therefore $prob(Z\ wins) = {}^{3}/_{20}$ and hence the odds against Z are 17 - 3.

2. The point P is a distance of 9 from the centre of a circle radius 15. How many different chords through P have integer length?

The longest chord is a diameter with length 30 and the shortest chord has length 24, since this is bisected at right-angles by the line length 9 from the centre and we have therefore a 3, 4, 5 Δ with the radius (enlarged by factor of 3). Chord lengths of 25, 26, 27, 28, 29 can all be achieved in two ways because the diagram is symmetrical about the diameter passing through P. The chords of length 24 and 30 can each be achieved in only one way. So the total number of chords of integer length through P is $1 + 2 \times 5 + 1 = 12$.

3. For what integer values of m is $\sqrt{m+\sqrt{m+\sqrt{m+...}}}$ *an integer?*

You need a way to tame an unwieldy expression like this, and one way is to give it a name and try to do something to simplify the situation, e.g. by finding a way of relating it to itself.
Here, letting $x = \sqrt{m+\sqrt{m+\sqrt{m+...}}}$, squaring gives $x^2 = m + x$, so for x to be an integer, write as a quadratic in the usual form $x^2 - x - m = 0$ and using the formula, $x = \dfrac{1 \pm \sqrt{1+4m}}{2}$ so for x to be an integer, $1 + 4m$ must be the square of an odd number, so $1 + 4m = (1 + 2k)^2 = 1 + 4k + 4k^2$, and hence $m = k(k + 1)$ for any integer k. So m is the product of any two consecutive integers (they could be negative but this does not add anything). i.e. 0, 2, 6, 12, 20, 30, etc. These are of course double the Triangle Numbers, which are given by $\frac{1}{2}n(n + 1)$ for any integer n.

4. Solve the simultaneous equations $\dfrac{3xy}{x+y} = 5,\quad \dfrac{yz}{y+z} = 4,\quad \dfrac{2zx}{z+x} = 3.$

I am not sure how many ways there might be to do this question, but one that occurs to me is to note that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{x+y}{xy}$ and this is very similar to the expressions given if we take their reciprocals.

The first gives $\dfrac{x+y}{xy} = \dfrac{3}{5}$, the second gives $\dfrac{y+z}{yz} = \dfrac{1}{4}$ and the third $\dfrac{z=x}{zx} = \dfrac{2}{3}$.

So we have $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{5}$, $\dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{4}$ and $\dfrac{1}{z} + \dfrac{1}{x} = \dfrac{2}{3}$, i.e simultaneous equations in $\dfrac{1}{x}$, $\dfrac{1}{y}$ and $\dfrac{1}{z}$.

We can now either subtract one pair and then add of the other or add all three to get $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{91}{120}$

and subtracting each pair in turn gives $\frac{1}{x}=\frac{61}{120}$, $\frac{1}{y}=\frac{3}{40}$ and $\frac{1}{z}=\frac{19}{120}$, so the values of x, y and z are 120/61, 40/3, and 120/19.

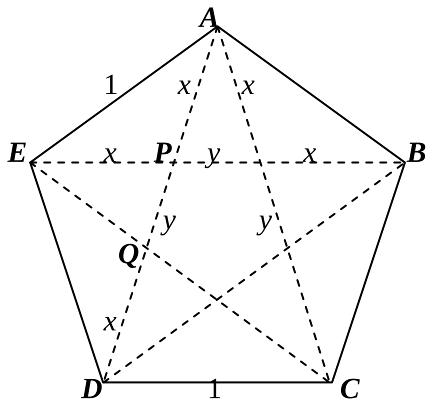
> *5. There are two circles that each pass through (1, 9) and (8, 8) which are tangential to the x-axis. Find the lengths of their radii.*

It is not at all obvious where the second circle is, but ignoring that (!), if you call the centre (a, b), then we have the distances from the two points are both equal to b, the distance from the x-axis. So we have

$$(8-a)^2+(8-b)^2=(1-a)^2+(9-b)^2=b^2.$$

Expanding and equating the first two of these, we find that $7a - b = 23$.
And $(8-a)^2+(8-b)^2=b^2$ leads to $a^2 - 16a - 16b + 128 = 0$. Substituting $b = 7a - 23$ gives a quadratic equation which factorises to $(a - 24)(a - 4) = 0$, so $a = 4$ or 124.
So $b =$ 5 or 845 !! The second circle I now realise has a centre way up to the right and will touch the x-axis at (124, 0). And the two radii are 5 and 845.

> *6. The five diagonals of a regular pentagon intersect at five points P, Q, R, S and T. Determine the ratio of the area of the pentagon PQRST to the area of the original pentagon.*



This diagram is full of interest. If we let the side of the original pentagon be 1, and $AP = x$ and $PQ = y$, then firstly we note that (as for any regular polygon), since this can be inscribed in a circle, then the angles at E between neighbouring diagonals are all equal (since they are subtended by equal chords), so $AEP = PEQ = QED = 36°$ and from angles in a triangle, angle $EPD = PED = 72°$.
So ΔPDE is isosceles and therefore $x + y = 1$.
And since $\Delta EPQ \sim \Delta EBC$, we have $x / y = (2x + y) / 1$.

But also since $\Delta AEP \sim \Delta ADE$, we have $(2x + y) / 1 = 1 / x$.
Hence $x / y = 1 / x$ and hence $y = x^2$.
Now since $x + y = 1$, we have $x^2 + x - 1 = 0$,
and so $x=\frac{-1\pm\sqrt{1+4}}{2}$ and since $x > 0$, $x = (\sqrt{5} - 1) / 2$.

And $y = 1 - x = (3 - \sqrt{5}) / 2$ and also the length of a diagonal is the Golden Ratio $(\sqrt{5} + 1) / 2$ as is well known. The answer to the question here is that the ratio of the areas is $y^2 : 1$
and $y^2 = (3 - \sqrt{5})^2 / 4 = (9 - 6\sqrt{5} + 5) / 4 = (14 - 6\sqrt{5}) / 4 = (7 - 3\sqrt{5}) / 2$.

> *7. If $xy + x + y = 71$ and $x^2y + xy^2 = 880$, find all possible values of $x^2 + y^2$.*

I expect there are several ways to go about this. The first to occur to me is to involve $x^2y + xy^2$ and the first equation by multiplying up by x and then y and then adding.

If you do this, you get $x^2y + x^2 + xy = 71x$
and $xy^2 + xy + y^2 = 71y$

Adding then gives $880 + x^2 + 2xy + y^2 = 71x + 71y$ which can be rewritten as
$880 + (x + y)^2 = 71(x + y)$ and hence
$(x + y)^2 - 71(x + y) + 880 = 0$.

So we can now treat this as a quadratic in $(x + y)$ and factorise to give
$(x + y - 16)(x + y - 55) = 0$.

So $x + y$ is either 16 or 55. Since $xy(x + y) = 880$, we see that xy is either 55 or 16.
Since $x^2 + y^2 = (x + y)^2 - 2xy$, $x^2 + y^2$ is either $16^2 - 2 \times 55$ or $55^2 - 2 \times 16$, i.e. 146 or 2993.

8. *The "AM - GM inequality" states that for any set of positive numbers* $x_{1,}x_2,\ldots,x_n$ *the arithmetic mean,* $\dfrac{x_1 + x_2 + \ldots + x_n}{n}$ $\geq$ *the geometric mean,* $\sqrt[n]{x_1 x_2 \ldots x_n}$.

a) *Prove that this is true for n = 2, i.e. that for a, b > 0,* $\dfrac{a+b}{2} \geq \sqrt{ab}$.

b) *Prove that this is true for n = 4, i.e. that for a, b, c, d > 0,* $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

c) *Can you find a way to use this to prove this is true for any set* $x_{1,}x_2,\ldots,x_n > 0$ *?*

a) Since $a, b > 0$, we can let $a = x^2$ and $b = y^2$.
So now we wish to prove that $(x^2 + y^2)/2 \geq xy$, i.e. that $x^2 + y^2 \geq 2xy$.
But since $(x - y)^2 \geq 0$ for all x, y, we have that $x^2 - 2xy + y^2 \geq 0$, and hence $x^2 + y^2 \geq 2xy$ as required.
This proves that the AM - GM inequality holds for $n = 2$.
Note that equality will occur when the square is zero, i.e $x = y$, i.e $a = b$.

b) By part (a), we know that $\dfrac{a+b}{2} + \dfrac{c+d}{2} \geq \sqrt{ab} + \sqrt{cd}$

but we can also use the same result to deduce that $\dfrac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt{\sqrt{ab} \times \sqrt{cd}}$ which is $\sqrt[4]{abcd}$.

Thus $\dfrac{a+b+c+d}{4} \geq \dfrac{\sqrt{ab} + \sqrt{cd}}{2} \geq \sqrt[4]{abcd}$. This proves the AM - GM inequality in the case $n = 4$.
Equality occurs when $a = b$ and $c = d$ (for the first step) and $ab = cd$ for the second step, so equality occurs when all the quantities a, b, c, d are equal.

c) We can continue this process to prove the result in the cases $n = 8, 16, 32$, etc.
Technically this is by "Induction", but for those who have not heard of this, you can see that the process above can be repeated in the same way. So now we just have to prove the result for values of n which are not powers of 2. Here is one way of proceeding:
I will give an example first in the case $n = 5$.
We have proved that the inequality is true for $n = 4$ and $n = 8$.
So for 5 values, say a, b, c, d, e, let the Arithmetic Mean be M, so we have $M = (a + b + c + d + e)/5$.
Now consider the **eight** values a, b, c, d, e, M, M, M.

Because we have proved the result for 8 values, we have $\dfrac{a+b+c+d+e+3M}{8} \geq \sqrt[8]{abcdeM^3}$

$$\Rightarrow a+b+c+d+e+\tfrac{3}{5}(a+b+c+d+e) \geq 8 \times \sqrt[8]{abcdeM^3}$$

$$\Rightarrow \tfrac{8}{5}(a+b+c+d+e) \geq 8 \times \sqrt[8]{abcdeM^3}$$

$$\Rightarrow M \geq \sqrt[8]{abcdeM^3} \qquad \Rightarrow M^8 \geq abcdeM^3 \qquad \Rightarrow M^5 \geq abcde$$

$$\Rightarrow M \geq \sqrt[5]{abcde} \qquad \Rightarrow \frac{a+b+c+d+e}{5} \geq \sqrt[5]{abcde},$$ as required.

For a formal proof, this method just needs to be written up generally, but the principle would be the same.

Supported by Man Group plc Charitable Trust