UKMT MENTORING SCHEME (Intermediate Level)
February 2012 (Sheet 5)
Questions

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1. Horses X, Y, Z are to run a 3-horse race (in which there are no dead heats). The odds against X winning are 3 - 1, and the odds against Y winning are 2 - 3. What are the odds against Z winning? [*Note: If the odds against a horse H winning are p - q, this means that p : q is the ratio prob(H does not win): prob(H does win).*]

2. The point P is a distance of 9 from the centre of a circle radius 15. How many different chords through P have integer length?

3. For what integer values of m is $\sqrt{m+\sqrt{m+\sqrt{m+\ldots}}}$ an integer?

4. Solve the simultaneous equations $\frac{3xy}{x+y}=5, \quad \frac{yz}{y+z}=4, \quad \frac{2zx}{z+x}=3.$

5. There are two circles that each pass through (1, 9) and (8, 8) which are tangential to the x-axis. Find the lengths of their radii.

6. The five diagonals of a regular pentagon intersect at five points P, Q, R, S and T. Determine the ratio of the area of the pentagon $PQRST$ to the area of the original pentagon.

7. If $xy + x + y = 71$ and $x^2y + xy^2 = 880$, find all possible values of $x^2 + y^2$.

8. The "AM - GM inequality" states that for any set of positive numbers $x_1, x_2, \ldots, x_n$ the arithmetic mean, $\frac{x_1+x_2+\ldots+x_n}{n}$ $\geq$ the geometric mean, $\sqrt[n]{x_1x_2\ldots x_n}$.

   a) Prove that this is true for $n = 2$, i.e. that for $a, b > 0$, $\frac{a+b}{2} \geq \sqrt{ab}$.

   b) Prove that this is true for $n = 4$, i.e. that for $a, b, c, d > 0$, $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$.

   c) Can you find a way to use this to prove this is true for any set $x_1, x_2, \ldots, x_n > 0$?

Deadline for receipt of solutions: 28th February 2012

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