BMOS MENTORING SCHEME (Intermediate Level)
January 2012 (Sheet 4)
Comments for Mentors and Teachers

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1. A toy store ordered 18 large and 7 small bags of identical marbles. When they were delivered it was discovered that the bags had broken open with all the marbles loose in the container. How did the store clerks make up the bags with the proper number of marbles in each, if the total number of marbles was 233?

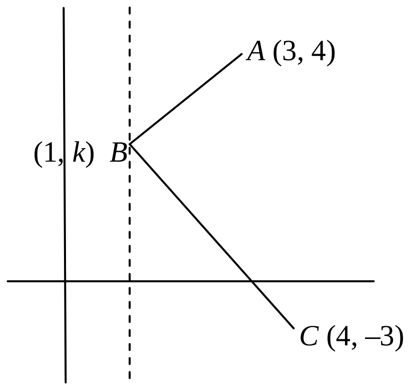
This question is really about divisibility. The temptation (which would work!) is just to use trial and error, but you learn nothing in the process. A better approach would be to look at multiples of 18 and 7. So we might look to see if 233 is near a multiple of 18. 2+3+3=8, so it is one less than a multiple of 9, and since 234 is even and a multiple of 9, it is divisible by 18. So 233 is 17 more than a multiple of 18. In the language of modular arithmetic, we would say $233 \equiv 17 \pmod{18}$. So we need a multiple of 7 which is equivalent to 17 mod 18. If we examine all the numbers $\equiv 17 \pmod{18}$, we get 17, 35, 53, ... and notice that $35 = 7 \times 5$.
So if we have 5 in a small bag, we have 35 marbles in small bags and need to make up 198 in large bags and since $198 = 18 \times 11$, that means the store clerks have to put 11 in each large bag and 5 in each small bag.

2. I didn't realize there was a power-cut during the night, so I leave to walk to the station as usual, when the electric clock in the hall (which is usually correct) says 8 o'clock. So I am surprised when I get to the station and the clock there says 9.30, which the ticket inspector assures me is the correct time. I catch the next train, and when I get back in the evening, as I set out for home, the station clock says 7.30. Of course, I'm tired now, so I can only walk at two-thirds of the pace I managed in the morning. When I get home, the hall clock says 7 o'clock. Assuming there hasn't been another power cut during the day, how long was the power off last night?

Suppose the power cut lasts x hours and my morning walk takes t hours.
We have $x + t = 1.5$ and $\frac{3}{2}t - x = -0.5$. Adding these gives $\frac{5}{2}t = 1$, so $t = 0.4$.
Therefore $x = 1.1$ and so the power cut lasted for 66 minutes.

3. The points A (3, 4), B (1, k) and C (4, –3) are three vertices of a rectangle ABCD. Find all possible values of k.

I suggest the first thing to do is to draw a diagram. A and C can be plotted precisely, and B we know lies on the line $x = 1$.
Two approaches suggest themselves:
1) Since the lines are perpendicular, the gradients multiply to –1. So we have:



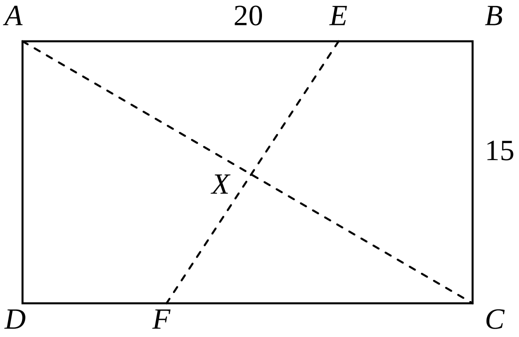
$$\frac{4-k}{3-1} \times \frac{-3-k}{4-1} = -1$$

So $(4 - k)(3 + k) = 6$
Thus $k^2 - k - 6 = 0$
So $(k - 3)(k + 2) = 0$ Hence $k = 2$ or -3.

2) Alternatively, using Pythagoras we have $(4 - k)^2 + 2^2 + (k + 3)^2 + 3^2 = 1^2 + 7^2$.
This gives $16 - 8k + k^2 + 4 + k^2 + 6k + 9 + 9 = 50$, so $2k^2 - 2k - 12 = 0$ leading to the same answers as before.

4.	*A rectangular piece of card ABCD measures 15 cm by 20 cm. The card is folded so that A folds exactly onto the opposite corner C. Calculate the length of the crease.*

This is a great question! Approachable and yet in some ways hard. Because there are certain things which are implicit in the folding process which you need to realise. It is quite helpful actually to get a piece of paper and fold it and start observing things.



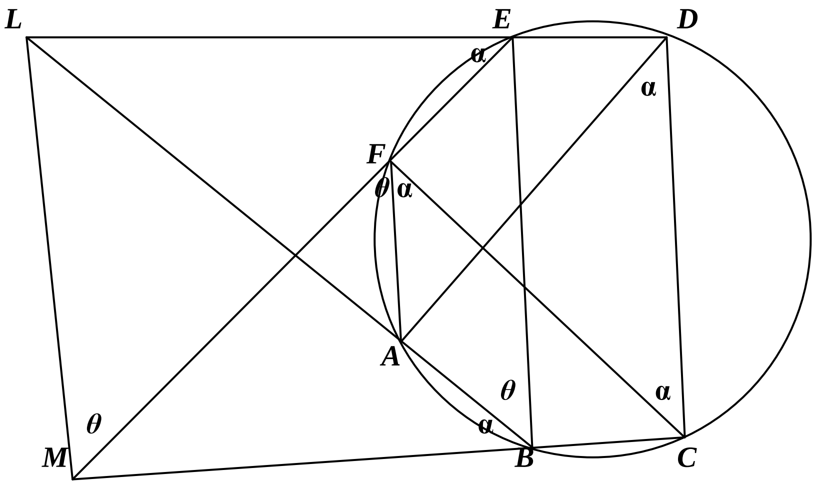
Suppose the crease is denoted by EF.
The key idea is that the diagonal AC is bisected at 90° by EF. So $\Delta AEX \sim \Delta ACB$, and by Pythagoras we know that $AC = 25$, so $AX = 25/2$.
By similar triangles, $EX/AX = BC/AB = 3/4$.
So $EX = 25/2 \times 3/4 = 75/8$.
Hence the length of the crease is 75/4 = 18.75 cm.

5.	*Solve the system of equations*	$x^2 - xy + y^2 = 3$ $x^2 - y^2 = 3.$

I think you need to mess around with this and see if you come up with anything. There are really only two things which occur to me, and that is either to add or subtract the equations. Normally one would not seek to get rid of the 3; however, on this occasion if you subtract this is helpful. We get $2y^2 - xy = 0$ which gives $y(2y - x) = 0$ so either $y = 0$ or $x = 2y$.
If $y = 0$, then $x^2 = 3$ and $x = \pm\sqrt{3}$. If $x = 2y$, then from the second equation we get $3y^2 = 3$, so $y = \pm 1$.
So the four solutions are $(\sqrt{3}, 0)$, $(-\sqrt{3}, 0)$, $(2, 1)$ and $(-2, -1)$.

It is important that you realise that you should check these in the original equations, since we only substituted back into one of the equations so the steps are not reversible (though actually both would have given the same in each case). However, these four solution pairs do all work. If you had known there were definitely four distinct solutions then that would be different. You do not. If you know about conic sections, you will know that since these are both second degree equations they are the intersections of two conics. In this case the first is an ellipse and the second is a hyperbola, but of course these do not necessarily intersect at 4 points, though it certainly will be at most four. [A "conic" is a section of a cone, which is rather like an old-fashioned egg-timer, with two bits both of which go off to infinity. If you slice it in different ways you get one of the three main conic sections: an ellipse, a parabola or a hyperbola. As stated above, the equations of these in x and y all are "second degree" which means that the highest term is a quadratic (either x^2 or xy or y^2).]

6. *ABCDEF is a cyclic hexagon in which AF is parallel to CD. BC meets EF at M and BA meets DE at L. Prove that LM is parallel to DC.*



Firstly, it is quite difficult to draw the diagram. After drawing the circle, I suggest it is probably best to choose suitable points for *L* and *M* and then draw lines back from these to produce suitable chords.
Now all that is needed is alternate angles, angles in the same segment and the exterior angle of a cyclic quadrilateral, although as is often the case it is difficult to see how to apply them.

Secondly, it is useful to have some sort of strategy. What might we aim to do to prove that *LM* is parallel to *FA?* Probably the most obvious thing to do is to try to prove that *LMF* = *MFA*, since these would be alternate angles and so *LM* would be parallel to *FA*.

Let us start with what we know. *AF* is parallel to *DC*. To use this, it is natural to draw chords *FC* and *AD*.
So we have *AFC* = *FCD* (alt. angles) = *LEF* (ext angle of cyc quad) = α.
Similarly *AFC* = *ADC* (angles in same seg) = *ABM* (ext angle of cyc quad).
So we now have *LBM* = *LEM* and therefore *LMBE* is cyclic. So draw in *BE* as well.
Hence *LME* = *LBE* (angles in same seg) = *MFA* (ext angle of cyc quad).
Since *LMF* = *MFA*, we have *LM* parallel to *FA*.

7. *Find all pairs of non-negative integers (m, n) which are solutions to the equation*

$$3(2^m)+1=n^2.$$

It is not immediately obvious what to do with this. So as always you need to play around with it. With Number Theory questions, we are often looking at divisibility and factorisation, so maybe this prompts the observation that if you take 1 from both sides, you get a difference of two squares. You should do this and then consider factors.
So, doing this we get $3(2^m) = (n+1)(n-1)$, and hence either $n + 1$ or $n - 1$ must be divisible by 3.
Suppose $n + 1 = 3s$. We then get $2^m = s(3s - 2)$, so both s and $3s - 2$ must be powers of 2.
If $s = 1$, then $3s - 2 = 1$ also, giving $n = 2$ and hence $2^m = 1$, so $m = 0$.
If $s = 2$, then $3s - 2 = 4$, giving $n = 5$, so $2^m = 8$ and hence $m = 3$.
If $s = 2^t$, where $t > 1$, then $3s - 2 = 3 \times 2^t - 2 = 2(3 \times 2^{t-1} - 1)$ and since 2^{t-1} is always even, the bracket is always odd, so this cannot be a power of 2.
Now suppose $n - 1 = 3s$. We now have $2^m = s(3s + 2)$.
If $s = 2$, this gives $3s + 2 = 8$, giving $n = 7$ and $2^m = 16$, so $m = 4$.
If $s = 2^t$, where $t > 1$, then $3s + 2 = 3 \times 2^t + 2 = 2(3 \times 2^{t-1} + 1)$ and since 2^{t-1} is always even, the bracket is always odd, so this cannot be a power of 2.
So we have a total of 3 solution pairs, namely (0, 2), (3, 5) and (4, 7).

8.	*Which positive integer less than 2012 has the most factors?* *When (i.e. which year after 2012) will this answer next be equalled?* *When will it lose its Gold Medal position?*

Firstly you need to know / understand / or work out, that to find the number of positive factors of any integer, they should express it as the product of primes and then consider the possible number of times to include each prime factor to get all possible factors.
e.g. To get the factors of $N = 2^3 \times 3^5 \times 7$, each possible divisor of N includes 2 to the power of either 0, 1, 2 or 3; 3 to the power of either 0, 1, 2, 3, 4 or 5; and 7 to the power of either 0 or 1.
So the number of (positive) factors is $4 \times 6 \times 2 = 48$.
If they don't know this idea, you could prompt them to discover it for themselves.
The general result is:

The number of factors of $2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} \times \ldots$ is $(a_1 + 1)(a_2 + 1)(a_3 + 1)(a_4 + 1)\ldots\ldots$

So if we are trying to maximise the number of factors, there is no point in using larger primes than necessary. Since $2 \times 3 \times 5 \times 7 \times 11 = 2310$, we do not need any primes higher than 7. The largest available powers are 2^{10}, 3^6, 5^4, and 7^3. We can now look at cases, classifying according to the power of 2. Note that there is never any point in using a power of 3 greater than the power of 2 etc. The best available such numbers are:

2^{10}	1024	11	11 divisors
$2^9 \times 3$	1536	10×2	20
$2^7 \times 3^2$	1152	8×3	24
$2^7 \times 3 \times 5$	1920	$8 \times 2 \times 2$	32
$2^6 \times 3^3$	1728	7×4	28
$2^5 \times 3^2 \times 5$	1440	$6 \times 3 \times 2$	36
$2^3 \times 3 \times 5 \times 7$	840	$4 \times 2 \times 2 \times 2$	32

So the winner is 1440 with 36 factors.
Despite the fact that this record has stood for 571 years, it will be equalled in just 4 years time, since $2016 = 2^5 \times 3^2 \times 7$ and hence has $6 \times 3 \times 2 = 36$ factors as well.
For 1440 to lose the Gold Medal position, we must wait another 509 years from now until 2520, since $2520 = 2^3 \times 3^2 \times 5 \times 7$, so it would have $4 \times 3 \times 2 \times 2 = 48$ factors.

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