BMOS MENTORING SCHEME (Intermediate Level)
December 2011 (Sheet 3)
Solutions and Explanatory Notes

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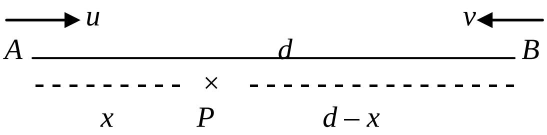
1. *A pedestrian started walking from A to B. At the same time a biker started cycling from B to A. When the biker met the pedestrian, he immediately picked him up, turned round and took him to B and then, without any delay went to A. As a result, the biker spent two and a half times longer cycling than if he had gone directly from B to A. If the pedestrian had not got a lift, how many times longer would he have spent going from A to B?*

I suggest you think of starting by introducing some letters and writing down some equations. A diagram is helpful to fix things in your mind. Here is a possible solution:



Suppose they meet at P, x km from A and that $AB = d$
The cyclist travels a dist of $d + 2(d - x) = 3d - 2x$ instead of d.

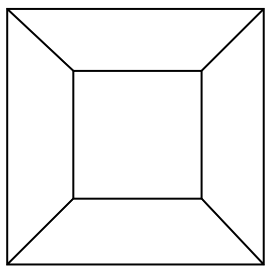
So if his speed is v, we have $\dfrac{3d-2x}{v} = \dfrac{5}{2} \times \dfrac{d}{v}$, giving $6d - 4x = 5d$ so $d = 4x$, so the cyclist travels $3x$.
whilst the pedestrian travels x, so the cyclist travels three times as fast. So $v = 3u$.

If the pedestrian's speed is u, the pedestrian's time is $\dfrac{x}{u} + \dfrac{d-x}{v} = \dfrac{x}{u} + \dfrac{d-x}{3u} = \dfrac{d+2x}{3u}$.

His time would have been $\dfrac{d}{u}$ so he takes k times as long, where $\dfrac{d}{u} = k \times \dfrac{d+2x}{3u}$.

So $3d = k(d + 2x)$ and hence $k = \dfrac{3d}{d+2x} = \dfrac{12x}{6x} = 2$. So the pedestrian would have taken twice as long.

2. *Is it possible to number the eight vertices of a cube from 1 to 8 in such a way that the value of the sum of the two ends of each edge is different from that of any other edge?*



Well, you have to start thinking about it and start getting into the problem.
A diagram would be useful, and a perspective drawing as shown is nice and clear in 2D.
If it is possible, the vertices are labelled 1 to 8, and the sums would have to have values from 1 + 2 to 7 + 8, i.e. from 3 to 15. This gives 15 – 3 + 1 = 13 possible values. There are 12 edges to consider. So it looks as if it might be possible. All the values would have to be used except one.
Now consider the small values: to get 3 and 4 we have to have 1 adjacent to both 2 and 3. Therefore 2 is not adjacent to 3, so the only way to get 5 is to have 1 adjacent to 4 as well. But then there is no way of getting 6, since 1 cannot be adjacent to 5, and from the above 2 is not adjacent to 4. So one of the totals 3 to 6 cannot be used.

Similarly for the large totals: to get 15 and 14 we have to have 8 adjacent to both 7 and 6. Therefore 7 is not adjacent to 6, so the only way to get 13 is to have 8 adjacent to 5 as well. But now there is no way of getting 12 since 8 cannot be adjacent to 4, and from the above 7 is not adjacent to 5. So one of the totals 12 to 15 cannot be used. This only leaves 11 possible totals, but we have 12 edges, and by the Pigeon-hole Principle you cannot have 12 different edge totals if there are only 11 numbers available.

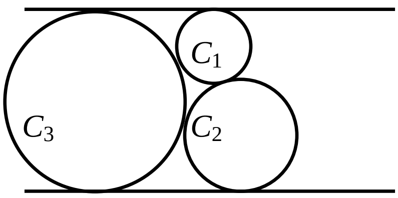
An alternative approach can be used as follows:
The sum of the 12 edges must be 3× (1 + 2 + ... + 8), since every vertex is at the end of three edges.
This is 108, so the sum of the 12 edges has to be 108, but the sum of the integers from 3 to 15 is 117. Thus the one value which must be excluded is 9. Then you need to apply one of the arguments above.

3.	*Find all possible pairs of positive integers whose sum and product add to 2011.*

We wish to find integers m and n such that $mn + m + n = 2011$.
Since we are working in whole numbers, it is useful to look for factorisations, but unfortunately the Left Hand Side will not factorise. On the other hand a simple addition will make it factorise.

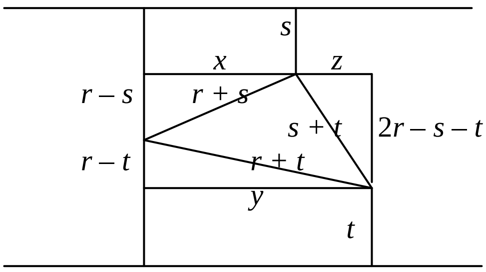
The number to add is 1, giving $mn + m + n + 1 = 2012$, which factorises to $(m + 1)(n + 1) = 2012$.
So we want two numbers to multiply to 2012, and we want all the possibilities for this.
So we need to consider the prime factorisation of $2012 = 2 \times 1006 = 2^2 \times 503$ and 503 is prime.
So the possible pairings for $(m + 1, n + 1)$ are (2, 1006) and (4, 503) since $m, n > 0$.
So subtracting 1 just gives two possible pairs: (1 and 1005) and (3 and 502).
Check: $1005 + 1005 + 1 = 2011$ and $3 + 502 + 1506 = 2011$.

4.	*Three circles C_1, C_2, C_3 are mutually touching and are also touching two parallel lines as shown. If the circles C_1 and C_2 have radii s and t, find the radius of C_3.*



If we reduce the clutter by just joining up the centres and creating a few right angled triangles, we have the lengths as shown in the diagram below.
Suppose the radius of circle C_3 is r.
The distance between the two parallel lines is $2r$.

Using Pythagoras' Theorem on the right-angled triangles in turn, we have:



$x^2 = (r + s)^2 - (r - s)^2 = 4rs$ so $x = 2\sqrt{(rs)}$
and $y^2 = (r + t)^2 - (r - t)^2 = 4rt$ so $y = 2\sqrt{(rt)}$

Since $z = y - x$,
$z = 2\,(\sqrt{(rs)} - \sqrt{(rt)})$

Now in the other right-angled triangle,
$z^2 = (s + t)^2 - (2r - s - t)^2 = 2r \times 2(s + t - r)$
So $4(rs - 2\sqrt{(r^2st)} + rt) = 4r\,(s + t - r)$
giving $rs - 2\sqrt{(r^2st)} + rt = rs + rt - r^2$
so $r^2 - 2r\sqrt{(st)} = 0$
$\Rightarrow$ $r\,(r - 2\sqrt{(st)}) = 0$. Since $r \neq 0$, we conclude that $r = 2\sqrt{st}$.

5.	*Solve the system of equations:*	$x^2 + x - 1 = y$ $y^2 + y - 1 = z$ $z^2 + z - 1 = x$

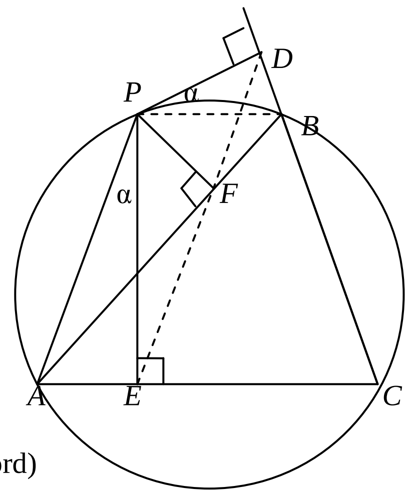
It isn't obvious what to do with equations like this. You could try adding them all together and then cancel the $x + y + z$ but that doesn't seem to lead very far. You could subtract (2) from (1) and use the $(x - y)$ factor on the left but that doesn't seem to lead very far either. After some thought you might come up with this:

If you add 1 to both sides of each equation and factorise, you get:

$$x(x+1) = y+1$$
$$y(y+1) = z+1$$
$$z(z+1) = x+1$$

Now, multiplying together gives $xyz(x+1)(y+1)(z+1) = (x+1)(y+1)(z+1)$,
so we have $(x+1)(y+1)(z+1)(xyz-1) = 0$ so either $x = -1$ or $y = -1$ or $z = -1$ or $xyz = 1$.
If $x = -1$, then $y = -1$ and $z = -1$, so one solution is (–1, –1, –1).
The other possibility is that $xyz = 1$.
If this is the case, either $x = y = z = 1$, which is a solution or one of the variables has magnitude greater than 1. Suppose that this is x. If $x > 1$, then from equation (1), $y > x$ and from (2), $z > y$ and from (3), $x > z$, which is impossible.
If however, $x < -1$, then because $x = z^2 + z - 1 = (z+1)^2 - \frac{5}{4}$, x can only be in the range $-\frac{5}{4} \le x < -1$, and hence $x + 1$ is in the range $-\frac{1}{4} \le x+1 < 0$ and since $y = x(x + 1) - 1$, y must be negative.
Then from equation (2), z would also have to be negative so xyz could not = 1.
Therefore the only solutions are (–1, –1, –1) and (1, 1, 1).

6.	*A point P is chosen on the circumcircle of the triangle ABC. Perpendiculars are dropped from P to the points D, E and F on sides BC, CA and AB respectively. Prove that the points D, E and F lie on a straight line.*



We wish to prove that *D*, *F* and *E* are collinear.
i.e. that $\angle DFB = \angle AFE$.

Firstly, note that we have several cyclic quadrilaterals: *AEFP*, *PFBD* and *PDCE* and of course *APBC*.

Proof

Since $\angle APB = 180 - C$ (opp $\angle$s of cyc quad)
and also $\angle EPD = 180 - C$ (opp $\angle$s of cyc quad)
we have $\angle APB = \angle EPD$,
and by subtracting the common part of both angles,
we have $\angle APE = \angle DPB = \alpha$ (say).
But *AEFP* is cyclic since $\angle AEP = \angle AFP$ ($\angle$s on same chord)
therefore $\angle AFE = \angle APE = \alpha$ ($\angle$s on same chord).
Similarly, *PDBF* is cyclic since $\angle PDB = \angle PFB = 90°$
therefore $\angle DPB = \angle DFB = \alpha$ ($\angle$s on same chord).
Hence $\angle DFB = \angle AFE$ as required, so *D*, *F* and *E* are collinear.

The key thing so often is to look for cyclic quadrilaterals, so you should get familiar with the standard circle theorems: angle in semicircle, same segment theorem, opposite angles of cyclic quadrilateral, alternate segment theorem, and hopefully the intersecting chords theorem (3 versions) and just get used to working with these and arguing carefully and logically. Even when circles are not explicitly mentioned, still look for these theorems. [A paper including all of these is included on the Mentoring Section of the UKMT website.]

You should also consider different possible configurations. It is helpful here to use a dynamic geometry package such as Cabri Gometre or Geometer's Sketchpad or Geogebra (free on-line!). I think you will find here that two of the feet of the perpendiculars are on the sides of the triangle and one on an extended side. So you ought to prove the result as well where *F* is not internal to *AB*. I will leave the proof of this as an exercise for you! (The case where *E* is external to *AC* and *D* internal to *BC* is essentially the same proof as above.)

7. The notation $|x|$ is defined as follows: $|x| = x$ if $x \geq 0$
$|x| = -x$ if $x < 0$

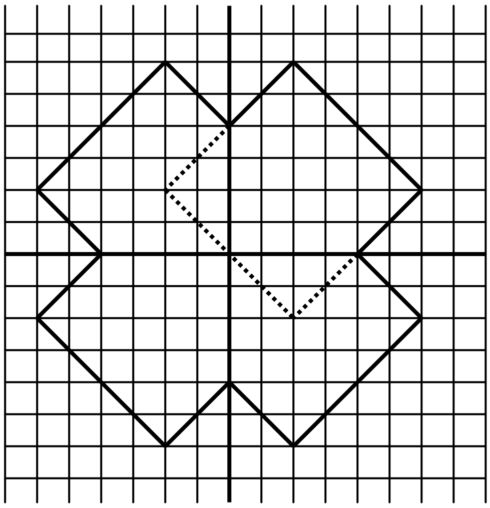
Find the area enclosed by the polygon whose equation is $\big||x|-2\big| + \big||y|-2\big| = 4$.

First you have to get your mind round the modulus function, and that is really the point of the question. So perhaps the least technical approach is just to start plotting some points, e.g. $x = 0 \Rightarrow |y| - 2 = \pm 4$, so $|y| = 6$ or -2, but -2 is impossible so $y = \pm 6$. We note incidentally that neither x nor y can be numerically greater than 6, so this reduces it to quite a manageable finite range of values.

Proceeding in this way, we get the following :

x	0	0	±1	±2	±3	±4	±4	±5	±5	±6
y	0	±4	±5	±6	±5	±4	0	±3	±1	±2

and the graph is



To find the area, one way is to add squares to the corners to get a large square meeting the axes at ±8. This square has side $8\sqrt{2}$, from which we must subtract 4 small squares of side $2\sqrt{2}$.
So the area is $(8\sqrt{2})^2 - 4 \times (2\sqrt{2})^2 = 128 - 4 \times 8 = 128 - 32 = 96$.

If you want a more sophisticated way of plotting the graph, you could start with the graph of $|x| - |y| = 4$, which is a square centred on the origin meeting the axes at ±4. To get the graph of $|x-2| - |y-2| = 4$, we translate by the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ getting the square which is partially dotted in the graph. Then because we are taking $|x|$ and $|y|$, to get the final graph, we take the portion of this square which lies in the 1st quadrant (the non-dotted part), and reflect it first in the y-axis (because we are using $|x|$), and then all this is reflected in the x-axis (because we are using $|y|$). Note that the final graph includes the isolated point (0, 0) (since this is in the portion of the square where $x \geq 0$ and $y \geq 0$, but this does not affect the overall area.

8.	*Write the expression* $\dfrac{1}{\sqrt{2011+\sqrt{2011^2-1}}}$ *in the form* $\sqrt{a}-\sqrt{b}$ *where a, b are integers.*

If we call the expression x then maybe it is natural to square to get rid of at least one root, so we have

$$x=\frac{1}{\sqrt{2011+\sqrt{2011^2-1}}}\Rightarrow x^2=\frac{1}{2011+\sqrt{2011^2-1}}$$

and after a bit of thought, maybe there are two things we might do, one being to take the reciprocal and the other being to multiply up by $2011-\sqrt{2011^2-1}$. The latter gives

$$x^2=\frac{1}{2011+\sqrt{2011^2-1}}\times\frac{2011-\sqrt{2011^2-1}}{2011-\sqrt{2011^2-1}}=\frac{2011-\sqrt{2011^2-1}}{2011^2-\left(2011^2-1\right)}=2011-\sqrt{2011^2-1}$$

Whilst the former gives $\dfrac{1}{x^2}=2011+\sqrt{2011^2-1}$

So adding these gives $x^2+\dfrac{1}{x^2}=4022$. A little more inspiration is still needed!

If we note that $\left(x+\dfrac{1}{x}\right)^2=x^2+\dfrac{1}{x^2}+2$ whereas $\left(x-\dfrac{1}{x}\right)^2=x^2+\dfrac{1}{x^2}-2$

we see that $\left(x+\dfrac{1}{x}\right)^2=4024$ and $\left(x-\dfrac{1}{x}\right)^2=4020$

Now clearly x is positive from its definition. but it is close to $\dfrac{1}{\sqrt{2011+2011}}=\dfrac{1}{\sqrt{4022}}\approx\dfrac{1}{60}$, i.e. small!

So $\dfrac{1}{x}>x$, so when we square root we have $x+\dfrac{1}{x}=\sqrt{4024}$ but $\dfrac{1}{x}-x=\sqrt{4020}$,

so subtracting these gives $2x=\sqrt{4024}-\sqrt{4020}$ and hence $x=\sqrt{1006}-\sqrt{1005}$.

I hope these comments are helpful and that your mentees enjoy doing the sheet. If you do have any comments either on the problems or the hints or the solutions which help me to target subsequent ones, a brief email would be great. Feedback to mentoring@ukmt.org is of course also very welcome.

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