BMOS MENTORING SCHEME (Intermediate Level)
November 2011 (Sheet 2)
Solutions & Explanatory Notes

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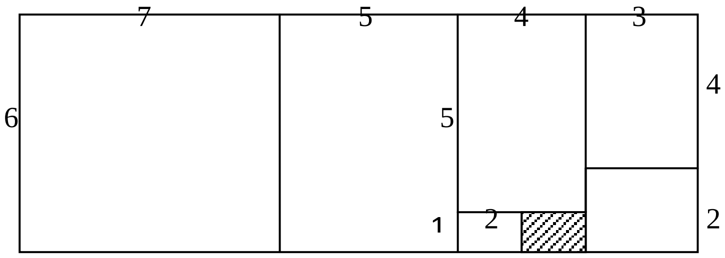
1. *A man whose birthday was January 1st died on February 9th 1992. He was x years of age in the year x^2 AD for some integer x. In what year was he born?*

Since the man dies in 1992, $x^2 < 1992$, so $x \leq 44$ since x is an integer.
Also he must have been $1992 + x - x^2$ when he died.
But if $x \leq 43$, he must have been at least $1992 - 43^2 + 43 = 186$ when he died, which is unrealistic, so we must have $x = 44$. So the man was 44 in 1936, so was born in 1892 and 100 when he died.

2. *You are given six rectangular tiles measuring 1 × 2, 2 × 3, 3 × 4, 4 × 5, 5 × 6, and 6 × 7 units. Find, with justification, the area of the smallest integer-sided rectangle into which these tiles can be fitted without overlap.*

Again, another nice non-standard problem which requires some thought and ingenuity. Firstly you might look at the total area of the rectangles which is 112. So the area of the best rectangle must be at least 112. If it is exactly 112, since 112 = 2 × 7 × 8, it must be either 7 × 16 or 8 × 14. If one places the 6 × 7 into either of these rectangles, you will find that it leaves a strip which cannot be completely filled so neither possibility works. So we need to try larger rectangles. Since 113 is prime, this clearly will not work. So we need to try 114 which = 2 × 3 × 19, so the rectangle would need to be 6 × 19. This can be made to work, and of course you need to demonstrate how this can be done. One way is:



[*You must of course prove this is optimal as above, i.e. prove that 112 and 113 are impossible.*]

3. *If p and $p^2 + 14$ are both prime numbers, find, with justification, all possible values of p.*

The natural thing to do here is to start experimenting with small values of p and this is a good idea!
If $p = 2$, $p^2 + 14 = 18$; if $p = 3$, $p^2 + 14 = 23$; if $p = 5$, $p^2 + 14 = 39$; if $p = 7$, $p^2 + 14 = 63$;
if $p = 11$, $p^2 + 14 = 135$; if $p = 13$, $p^2 + 14 = 183$.
So apart from $p = 3$, it is helpful to notice that $p^2 + 14$ is divisible by 3, and you should look for such a thing.
So it would be natural to conjecture that this might always be the case.
If $p > 3$, then p is either 1 more or 1 less than a multiple of 3. i.e. $p = 3k \pm 1$ or in the notation of modular arithmetic, $p \equiv \pm 1 \pmod 3$ and hence $p^2 \equiv 1 \pmod 3$. Hence $p^2 + 14 \equiv 0 \pmod 3$.
[$p^2 = (3k \pm 1)^2 = 9k^2 \pm 6k + 1$ yields the same thing - that p^2 is 1 more than a multiple of 3, so $p^2 + 14$ must be a multiple of 3.]
Either way, we see that for primes $p > 3$, $p^2 + 14$ is always a multiple of 3, so never prime.
Hence the only value of p which works is $p = 3$.

[Note: Modular arithmetic is a very useful weapon in Number Theory. There is a very helpful article on this on the Mentoring Section of the UKMT web site.]

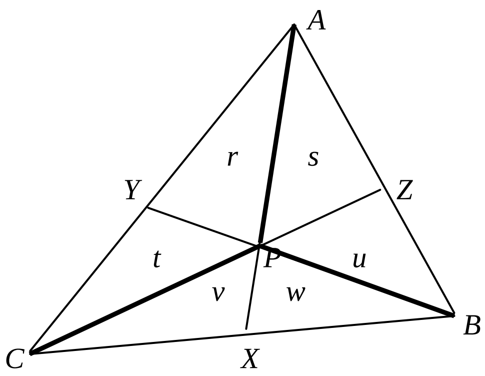
4. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$, find, with justification, the maximum possible value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ if a, b and c are positive integers.

The first thing to do here again is to start experimenting by trying values of *a, b, c* (**without a calculator!!**). We soon see that we don't want large values of *a, b, c* - we want them as small as possible, but clearly (*a, b, c*) = (2, 2, 2) gives a sum >1, as does (2, 2, 3), and (3, 3, 3) gives 1 which is not allowed. So unless we use a 2, the best we can do is (3, 3, 4) which gives 11/12. Now we need to investigate what happens with a 2. It is probably helpful to use the fact that we can specify that $a \le b \le c$, since the order is irrelevant, and the phrase to use here is "without loss of generality, we can assume that..." (this is often shortened to w.l.o.g.!)

So w.l.o.g assume $a \le b \le c$, and take $a = 2$. b cannot be 2 since ½ + ½ = 1, so suppose $b = 3$.
In this case the highest that c can be is 7 since 1/2 + 1/3 + 1/6 = 1, and we have 1/2 + 1/3 + 1/7 = 41/42.
Now we just have to deal with the case when $a = 2$, $b > 3$.
The best we can do here is (2, 4, 5) giving a sum of 19/20, which is less than 41/42, so this is the maximum.

5. Let ABC be a triangle and P be a point which lies inside it. Let AP meet BC at X, BP meet CA at Y, and CP meet AB at Z. If the point P divides each of the lines AX, BY and CZ in the same ratio, find this ratio. [i.e. If AP : PX = BP : PY = CP : PZ = k, find k.]

Q5. If $AP : PX = BP : PY = CP : PZ = k : 1$,
we can see that the three triangles highlighted, i.e.
Δs *APB*, *BPC* and *CPA* are all $1/(k+1)$ of Δ*ABC*.
So $3/(k+1) = 1$ and hence $k = 2$.
So *P* is a point of trisection of the Cevians *AX*, *BY* and *CZ*.
[This is all using the fact that the heights of Δs *CPB* and *CAB* are in the proportion *XP* : *XA* and similarly for the other triangles.]

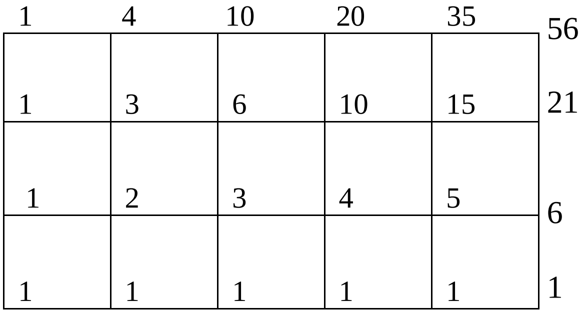


Note that if we denote the 6 small Δ areas by *r, s, t, u, v, w* as shown, we have $r + t = s + u = v + w$.
But we also now have $[CPX] = ⅓\,[CAX]$,
so $r + t = 2v$ and hence $v = w$ so since Δs *CPX* and *BPX* have the same area and height, they must have the same base, so *CX* = *BX*¸ and hence the Cevian *AX* is in fact a median of Δ*ABC*.
Similarly for the others, and *P* is of course the intersection of the medians, i.e. the centroid of the triangle.
[Note also that this proves that the centroid is the only point inside the triangle which divides all three Cevians passing through it in the same ratio.]

6.	*a)*	*How many routes of length 8 are there from (0, 0) to (5, 3) moving just along the grid lines?*
	b)	*A strange dart board has just 5 regions with associated scores of 1, 4, 16, 64, 256. How many scores are possible with exactly three darts? (Any dart may score zero.)*
	c)	*Explain the connection between (a) and (b).*

a) If we denote moves to the right and up by R and U respectively, this is the number of permutations of RRRRRUUU and this can be done is ${}^8C_5$ ways which is $8!/(5!\times3!) = 8 \times 7 \times 6/6 = 56$ ways.

You could also arrive at this by considering the number of paths to points along the way and noting that to get to some point (a, b), you have to come immediately from either $(a - 1, b)$ or from $(a, b - 1)$, so the number of paths to (a, b) can be found by adding the number of paths to each of these two points and build up as shown.

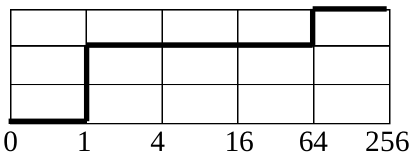


This is exactly how Pascal's Triangle is constructed, and the addition property above can be couched in terms of Binomial coefficients as either ${}^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1}$ or using the other notation $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$.

b) First note that if 3 darts score any of 0, 1, 4, 16, 64, 256 each then all the totals are distinct, so you could say that either all the darts are the same (${}^6C_1 = 6$ ways) or they are all different (${}^6C_3 = 20$ ways) or two are the same and one different (${}^6P_2 = {}^6C_2 \times 2 = 30$ ways), so a total of 56 ways.

c) We note that the answers to (a) and (b) are the same, so the question is why. One answer is that in part (b) we have ${}^6C_1 + {}^6C_2 + {}^6C_2 + {}^6C_3$ and since ${}^6C_1 + {}^6C_2 = {}^7C_2$ and ${}^6C_2 + {}^6C_3 = {}^7C_3$, we have ${}^7C_2 + {}^7C_3 = {}^8C_3 = 56$ ways as before. But it would be nice to feel that we could link it up with paths on a grid like the grid from (0, 0) to (5, 3).

We can do this as follows:



As shown, label the vertical lines with the scores. Now any path from (0, 0) to (5, 3) will represent one of the scores. So the path shown represents 1 + 1 + 64 = score of 66. Since each path represents exactly one of the scores, the number of possible scores is ${}^8C_3$ (choosing which of the 8 steps to go up) - or of course ${}^8C_5$.

7. a) *A well-known challenge is to find expressions equal to all the integers from 1 to 20 using* ***precisely four*** *4's and any number of the symbols +, –, ×, ÷, !, √, (,). So for example 7 = 4 × √4 – 4 ÷ 4. (Note you cannot use ³√ etc because of the 3.) Try this.*

b) *Now try to make the numbers 1 to 20 using* ***up to five*** *π's. You may also use the floor function ⌊ ⌋ where ⌊x⌋ means the greatest integer less than or equal to x, and the ceiling function ⌈ ⌉ where ⌈x⌉ means the smallest integer greater than or equal to x.*

a) Examples are: $1 = \frac{44}{44}$, $2 = \frac{4}{4} + \frac{4}{4}$, $3 = \frac{4+4+4}{4}$, $4 = \frac{4}{\sqrt{4}} + \frac{4}{\sqrt{4}}$, $5 = \sqrt{4} + \sqrt{4} + \frac{4}{4}$,

$6 = \frac{4!}{4} + 4 - 4$, $13 = \frac{44}{4} + \sqrt{4}$, $17 = 4^{\sqrt{4}} + \frac{4}{4}$. I will leave the rest for you to complete!

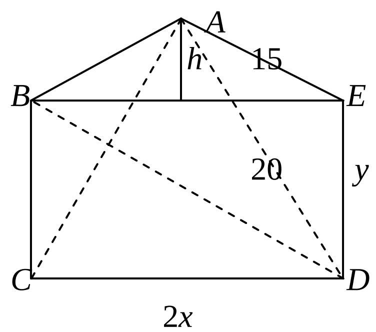
b) I suppose you could amuse yourself by doing this using the fewest number of π's in each case.

Some examples would be: $1 = \lfloor \sqrt{\pi} \rfloor$, $5 = \frac{\pi + \pi}{\pi} + \lfloor \pi \rfloor$ and $11 = \left\lfloor \frac{\pi^{\pi}}{\pi} \right\rfloor$.

8. *The figure shows a Heron pentagon in which* ***the sides, the diagonals and the area are all integers****. In addition, AB = AE and BCDE is a rectangle.*

a) If AB = 15 and AC = 20, find the area & length BD.

b) Give a set of general expressions for the sides, the diagonals and the area to generate an infinite family of such Heron pentagons.



a) If we introduce variables x, y, h as shown, we have $h^2 + x^2 = 15^2$ and $(h + y)^2 + x^2 = 20^2$, which implies that $y(2h + y) = 175 = 5^2 \times 7 > y^2$. Thus since $x > 0$, we must have $y = 7$, $h = 9$ and $x = 12$. This yields $BD^2 = 4x^2 + y^2 = 625$, giving $BD = 25$. This gives an area of 276, satisfying the condition that the area is an integer.

b) Here is one such set of values:

Set	AB	=	AE	=	$(m^2 - n^2)(m^2 + n^2)$
	BC	=	DE	=	$6m^2n^2 - m^4 - n^4$
	CD	=	BE	=	$4mn(m^2 - n^2)$
	AC	=	AD	=	$2mn(m^2 + n^2)$
	BD	=	CE	=	$(m^2 + n^2)^2$

Then the area $[ABCDE] = 2mn(m^2 - n^2)(10m^2n^2 - m^4 - n^4)$,

where m, n are coprime and $n < m < (\sqrt{2} + 1)n$.

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