UKMT MENTORING SCHEME (Intermediate Level)
November 2011 (Sheet 2)
Questions

Here is the second sheet of problems for this year. Again they are graded, the earlier questions being easier. Enjoy!

1. A man whose birthday was January 1st died on February 9th 1992. He was x years of age in the year x^2 AD for some integer x. In what year was he born?

2. You are given six rectangular tiles measuring 1×2, 2×3, 3×4, 4×5, 5×6, and 6×7 units. Find, with justification, the area of the smallest integer-sided rectangle into which these tiles can be fitted without overlap.

3. If p and $p^2 + 14$ are both prime numbers, find, with justification, all possible values of p.

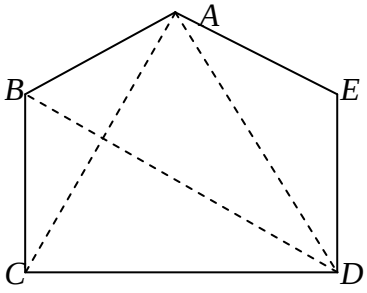
4. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$, find, with justification, the maximum possible value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ if a, b and c are positive integers.

5. Let ABC be a triangle and P be a point which lies inside it. Let AP meet BC at X, BP meet CA at Y, and CP meet AB at Z. If the point P divides each of the lines AX, BY and CZ in the same ratio, find this ratio. [i.e. If $AP : PX = BP : PY = CP : PZ = k$, find k.]

6. a) How many routes of length 8 are there from (0, 0) to (5, 3) moving just along the grid lines?
   b) A strange dart board has just 5 regions with associated scores of 1, 4, 16, 64, 256. How many scores are possible with exactly three darts? (Any dart may score zero.)
   c) Explain the connection between (a) and (b).

7. a) A well-known challenge is to find expressions equal to all the integers from 1 to 20 using **precisely four** 4's and any number of the symbols +, –, ×, ÷, !, √, (,). So for example $7 = 4 \times \sqrt{4} - 4 \div 4$. (Note you cannot use $\sqrt[3]{}$ etc because of the 3.) Try this.
   b) Now try to make the numbers 1 to 20 using **up to five** π's. You may also use the floor function $\lfloor\ \rfloor$ where $\lfloor x \rfloor$ means the greatest integer less than or equal to x, and the ceiling function $\lceil\ \rceil$ where $\lceil x \rceil$ means the smallest integer greater than or equal to x.

8. The figure shows a Heron pentagon in which **the sides, the diagonals and the area are all integers**. In addition, $AB = AE$ and $BCDE$ is a rectangle.
   a) If $AB = 15$ and $AC = 20$, find the area & length BD.
   b) Give a set of general expressions for the sides, the diagonals and the area to generate an infinite family of such Heron pentagons.



Deadline for receipt of solutions: 29th November 2011

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