BMOS MENTORING SCHEME (Intermediate Level)
October 2011 (Sheet 1)
Solutions and Explanatory Notes

I hope you enjoyed tackling the first sheet of the year. The questions are designed to be challenging but not impossible. Hopefully you will have found the first few quite accessible and then gradually they will have become more difficult. Here are some solutions, comments and notes which I hope you will find helpful.

1.	*Find the sum of the digits in the square of the number 111 111 111.*

If you start by writing out the long multiplication in the normal way you get:

```
          111111111
    ×     111111111
          111111111
         111111111
        111111111
      ...........
    111111111
    12345678987654321
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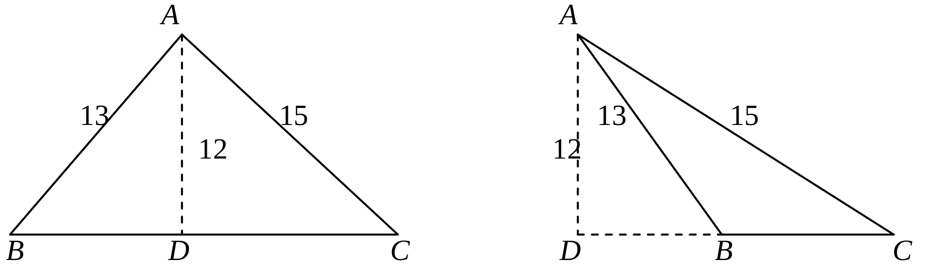
We see that the total in each column rises from 1 to 9 and back to 1, so the sum of all the digits is 81.

2.	*A train takes ¼ minute to pass a telegraph pole, and ¾ minute to pass through a tunnel 540 metres long. What is the length of the train?*

The way to approach this is to introduce some algebra. Natural letters to introduce are v m/s for the speed of the train, and something like l m for its length. So for the telegraph pole, we know it travels l m in 15 seconds, so $l = 15v$. And then for the tunnel $l + 540 = 45v$.
Solving simultaneously gives $30v = 540$, so $v = 18$ and there the length of the train, l, is 270 metres.

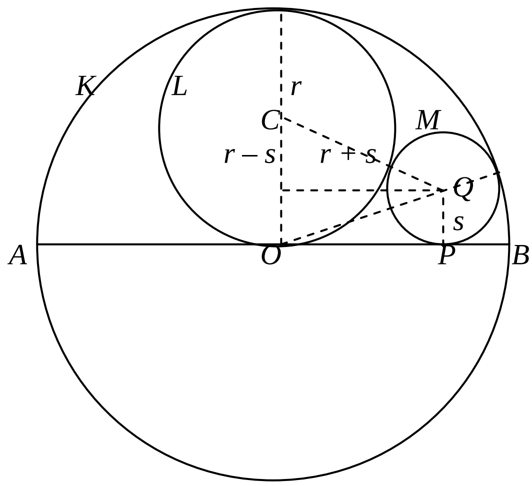
3.	*In ΔABC, the length of AB is 13cm, the length of AC is 15cm, and the length of the perpendicular from A to BC (i.e. the "altitude" from A) is 12cm. Find the two possible lengths of BC.*

With geometry problems it is important to consider all possible configurations. Here the triangle can either be acute angled, or $\angle ABC$ can be obtuse as shown.



In both cases we have BD = 5 cm (5, 12, 13 Δ) and DC = 9 cm (3, 4, 5 Δ enlarged × 3),
so the possible lengths for BC are either 9 + 5 = 14 cm, or 9 – 5 = 4cm.

4. *Circle K has diameter AB. Circle L touches K internally and also touches the line AB at the centre of circle K. Circle M touches L externally and K internally and also has tangent AB. Find the ratio of the area of circle K to the area of circle M.*



Q4. If we let the radius of L be r, and the radius of M be s, the radius of K will be $2r$.
So in the diagram $OQ = 2r - s$
and $CQ = r + s$.
Considering the length OP
in two different ways, we have
$OP^2 = (2r - s)^2 - s^2 = (r + s)^2 - (r - s)^2$.

So $4r^2 - 4rs = 4rs$
and hence $4r^2 = 8rs$
so $r = 2s$.

Hence the ratio
[Area of K] : [Area of M] $= (2r)^2 : s^2$
$= (4s)^2 : s^2 = 16 : 1$.

5. *How many odd numbers greater than 60000 can be made from the digits 5, 6, 7, 8, 9, 0 if no number contains any digit more than once?*

If we are going to make odd numbers over 60000 using the digits 5, 6, 7, 8, 9, 0 at max once each, we can have numbers with either 5 or 6 digits. We are also going to need to split the 5-digit ones:
5-digits beginning with 7 or 9: 2 choices for 1st digit, 2 choices for last digit, and $4 \times 3 \times 2$ for the others.
5-digits beginning with 6 or 8: 2 choices for 1st digit, 3 choices for last digit, and $4 \times 3 \times 2$ for the others.
6-digits: 3 choices for last digit, 4 choices for 1st digit, and $4 \times 3 \times 2 \times 1$ for the others.
So total = 96 + 144 + 288 = 528 different possibilities.

6. *The total area of all the faces of a cuboid is 22 cm^2, and the total length of all its edges is 24 cm. Find the length of any one of its internal diagonals.*

If we call the dimensions x, y, z, we have $2(xy + yz + zx) = 22$ so $xy + yz + zx = 11$.
And the sum of the edges is $4(x + y + z) = 24$, so $x + y + z = 6$.
Now we are looking for the length of a diagonal which (using Pythagoras twice) is $\sqrt{x^2 + y^2 + z^2}$.
Note that $(x + y + z)(x + y + z) = x^2 + y^2 + z^2 + 2(xy + yz + zx)$.
So we have $6^2 = x^2 + y^2 + z^2 + 22$, and hence $x^2 + y^2 + z^2 = 14$, so the length of a diagonal is $\sqrt{14}$.

7. *Let n be an integer greater than 6. Prove that if n – 1 and n + 1 are both prime, then $n^2(n^2 + 16)$ is divisible by 720. Is the converse true?*

Firstly, consider the prime factors of 720, i.e. $2^4 \times 3^2 \times 5$.
If we can show that $n^2(n^2 + 16)$ is divisible by 16 and also by 9 and also by 5, then it must be divisible by $16 \times 9 \times 5 = 720$. Note this is because these three factors are coprime.
So we consider divisibility by each of these in turn.

Firstly, divisibility by 16:
As $n - 1$ and $n + 1$ are both prime, n must be divisible by 2 (since $n > 6$). Thus n^2 is divisible by 4 and $(n^2 + 16)$ is also divisible by 4, and so $n^2(n^2 + 16)$ is divisible by 16.
Now, divisibility by 9:
One of n, $n - 1$ and $n + 1$ is divisible by 3. However, $n - 1$ and $n + 1$ are prime, so n must be divisible by 3 (again, since $n > 6$). Therefore n^2 is divisible by 9, so $n^2(n^2 + 16)$ is divisible by 9 also.
Now, divisibility by 5:
One of $n - 2$, $n - 1$, n, $n + 1$ and $n + 2$ is divisible by 5. However, $n - 1$ and $n + 1$ are prime, and $n > 6$, so it is neither of these. So either n is a multiple of 5, or one of $n - 2$ or $n + 2$ is divisible by 5.
If n is divisible by 5, then clearly n^2 is, so $n^2(n^2 + 16)$ is as well.
So now we have to consider the possibility that n is either 2 more or 2 less than a multiple of 5.
i.e. $n = 5k \pm 2$ for some integer k. In this case, $n^2 + 16 = (5k \pm 2)^2 + 16 = 25k^2 \pm 10k + 20$.
This is clearly divisible by 5 for both of these cases, so $n^2(n^2 + 16)$ is divisible by 5 in all cases.

Since $n^2(n^2 + 16)$ is divisible by 16, 9 and 5, it is divisible by $16 \times 9 \times 5 = 720$ since these factors are coprime.

Note that a nice way of writing the last bit about divisibility by 5 is to say that we know that $n \equiv \pm 2 \pmod 5$, so $n^2 \equiv 4$, and hence $n^2 + 16 \equiv 20 \equiv 0 \pmod 5$.
[We say that $n \equiv 2 \pmod 5$ i.e. "n is equivalent to 2 mod 5" if n has remainder 2 when divided by 5.]

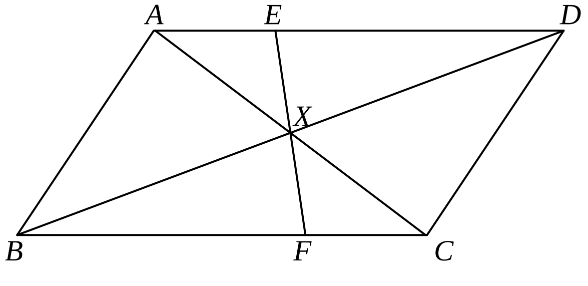
An alternative way of arguing the last bit about divisibility by 5 is to say:
$(n - 2)\, n\, (n + 2) = n^3 - 4n$ is a multiple of 5. Hence $n(n^3 - 4n)$ is also a multiple of 5.
Since $20n^2$ is clearly also divisible by 5, so also is $n^4 - 4n^2 + 20n^2$ which is equal to $n^2(n^2 + 16)$.

The converse, "If $n^2(n^2 + 16)$ is divisible by 720, then $n - 1$ and $n + 1$ are prime" is not true.
For example, if $n = 78$, then $78^2(78^2 + 16)$ is divisible by 720. However, 77 is not prime.
Or alternatively, if $n = 720$ then $720^2(720^2 + 16)$ is clearly divisible by 720 but 721 is divisible by 7.

8. *Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.*

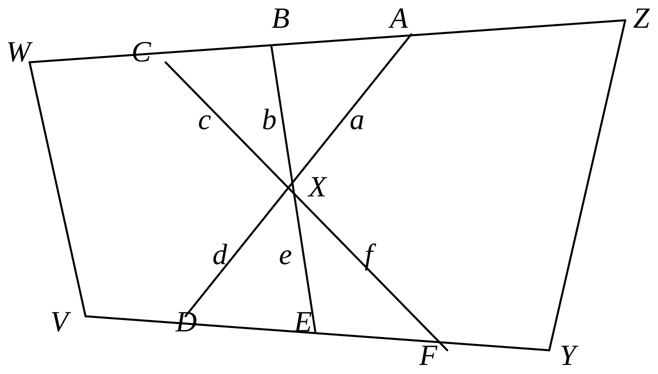
Firstly you need to appreciate that the "if and only if" means you have to prove two things:
Firstly, if G is a parallelogram then there is such a point, and secondly if there is such a point then G is a parallelogram.

Firstly the "if" part:
Suppose $ABCD$ is a parallelogram. Set X to be the point where the two diagonals cross. Let EF be some other line through X, where F lies on BC and E lies on AD. Without loss of generality, we can draw things in the configuration shown.



We must prove that the area of $ABFE$ is equal to the area of $CDEF$. But we know that the area of ABC is equal to the area of ADC, so we just need to show that the area of AXE is equal to the area of CXF.
Now, $AX = XC$ (the diagonals bisect each other), $\angle AXE = \angle CXF$ (vertically opposite angles) and $\angle EAX = \angle FCX$ (alternate angles). So in particular, they have the same area, as needed to be proved. Thus ΔAEX and ΔCFX are congruent (angle-side-angle). So in particular, they have the same area, as needed to be proved.

Secondly, the "only if" part:
Let G be a convex quadrilateral $WVYZ$, in which a point X exists such that all lines through X divide G into two equal areas, as in the diagram below.



The point X must be inside the quadrilateral, as otherwise we could draw a line through X, parallel to one of the sides, which does not intersect the quadrilateral at all, and this would clearly not divide it into equal areas.

Now, the area of $ADVW$ equals the area of $ADYZ$, and the area of $BEVW$ equals the area of $BEYZ$.

By removing the large common areas, this implies that the area of ABX is equal to the area of DEX, and thus that $\frac{1}{2}ab.\sin AXB = \frac{1}{2}de.\sin DXE$.
Also, $\angle AXB = \angle DXE$ (vertically opposite angles). Thus $ab = de$. Similarly, $bc = ef$ and $ac = df$.
Multiplying the first two of these, we get $ab^2c = de^2f$.
Then, dividing by the last of them, $b^2 = e^2$. Thus $b = e$. Similarly, we get $a = d$ and $c = f$.
Hence $\Delta AXB \cong \Delta DXE$ (side-angle-side). This means that $\angle XAB = \angle XDE$, so WZ is parallel to VY.
Similarly, WV is parallel to ZY, so $WVYZ$ is a parallelogram.

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